Journal of Approximation Theory **116**, 268–342 (2002)
doi:10.1006/jath.2002.3674

Classification Theorems for General Orthogonal Polynomials on the Unit Circle

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Communicated by Guillermo Lopez Lagomasino

Received August 28, 2000; accepted December 31, 2001

The set $\mathcal{P}$ of all probability measures σ on the unit circle $\mathbb{T}$ splits into three disjoint subsets depending on properties of the derived set of $\{|\varphi_n|^2\, d\sigma\}_{n\geqslant 0}$, denoted by $\mathrm{Lim}(\sigma)$. Here $\{\varphi_n\}_{n\geqslant 0}$ are orthogonal polynomials in $L^2(d\sigma)$. The first subset is the set of Rakhmanov measures, i.e., of $\sigma \in \mathcal{P}$ with $\{m\} = \mathrm{Lim}(\sigma)$, m being the normalized ($m(\mathbb{T}) = 1$) Lebesgue measure on $\mathbb{T}$. The second subset $\mathrm{Mar}(\mathbb{T})$ consists of Markoff measures, i.e., of $\sigma \in \mathcal{P}$ with $m \notin \mathrm{Lim}(\sigma)$, and is in fact the subject of study for the present paper. A measure σ, belongs to $\mathrm{Mar}(\mathbb{T})$ iff there are $\varepsilon > 0$ and $l > 0$ such that $\sup\{|a_{n+j}|: 0 \leqslant j \leqslant l\} > \varepsilon$, $n = 0, 1, 2, \ldots$, $\{a_n\}$ is the Geronimus parameters (= reflection coefficients) of σ. We use this equivalence to describe the asymptotic behavior of the zeros of the corresponding orthogonal polynomials (see Theorem G). The third subset consists of $\sigma \in \mathcal{P}$ with $\{m\} \subsetneq \mathrm{Lim}(\sigma)$. We show that σ is ratio asymptotic iff either σ is a Rakhmanov measure or σ satisfies the López condition (which implies $\sigma \in \mathrm{Mar}(\mathbb{T})$). Measures σ satisfying $\mathrm{Lim}(\sigma) = \{\nu\}$ (i.e., weakly asymptotic measures) are also classified. Either ν is the sum of equal point masses placed at the roots of $z^n = \lambda$, $\lambda \in \mathbb{T}$, $n = 1, 2, \ldots$, or ν is the equilibrium measure (with respect to the logarithmic kernel) for the inverse image under an m-preserving endomorphism $z \to z^n$, $n = 1, 2, \ldots$, of a closed arc J (including $J = \mathbb{T}$) with removed open concentric arc J_0 (including $J_0 = \varnothing$). Next, weakly asymptotic measures are completely described in terms of their Geronimus parameters. Finally, we obtain explicit formulae for the parameters of the equilibrium measures ν and show that these measures satisfy $\{\nu\} = \mathrm{Lim}(\nu)$.

1. INTRODUCTION

1.1. *Orthogonal polynomials and Schur's algorithm.* The Herglotz formula

$$F_\sigma \stackrel{\text{def}}{=} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma(\zeta) = \frac{1 + z f^\sigma(z)}{1 - z f^\sigma(z)} \tag{1.1}$$

determines a one-to-one correspondence

$$f^\sigma = \mathcal{H}(\sigma)$$

0021-9045/02 $35.00

between the set $\mathscr{P}$ of probability measures on the unit circle $\mathbb{T} \stackrel{\text{def}}{=} \{z \in \mathbb{C}: |z| = 1\}$ and the set $\mathscr{B}$ of all holomorphic mappings of the unit disc $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ into itself. In what follows we often drop the superscript σ in f^σ when σ is fixed.

Since $\mathscr{B}$ is the unit ball of the Hardy algebra H^∞ in $\mathbb{D}$, i.e., the algebra of all bounded and holomorphic functions in $\mathbb{D}$, the restriction of the $*$-weak topology of H^∞ to $\mathscr{B}$ is the topology of the pointwise convergence in $\mathbb{D}$. Similarly, we can consider the restriction to $\mathscr{P}$ of the $*$-weak topology of the Banach space $M(\mathbb{T})$ of all finite Borel measures with the variation norm. These operations turn $\mathscr{B}$ and $\mathscr{P}$ into compact metric spaces. It is easy to see that

$$\mathscr{H} : \mathscr{P} \to \mathscr{B}$$

is a homeomorphism of metric spaces.

Definition. The function $f^\sigma = \mathscr{H}(\sigma)$ is called the *Schur function* of σ.

The standard orthogonalization algorithm by Gram–Schmidt applied to the sequence $\{z^n\}_{n \geqslant 0}$ of mononomials in the Hilbert space $L^2(d\sigma)$ results in the sequence $\{\varphi_n\}_{n \geqslant 0}$ of orthogonal polynomials in $L^2(d\sigma)$:

$$\varphi_n(z) = k_n z^n + \cdots + \varphi_n(0), \qquad k_n > 0, \quad n = 0, 1, \ldots,$$

(1.2)

$$\int_{\mathbb{T}} \varphi_i \overline{\varphi_j}\, d\sigma = \delta_{ij}, \qquad i, j = 0, 1, \ldots .$$

The number of polynomials φ_n obtained coincides with the dimension $\dim L^2(d\sigma)$ of $L^2(d\sigma)$ which equals the number of the growth points of σ.

Let σ be fixed and $f = f^\sigma = \mathscr{H}(\sigma)$ be the Schur function of σ. Since f is in $\mathscr{B}$, we can associate with σ a sequence of *Schur iterates*:

(1.3)

$$f(z) \stackrel{\text{def}}{=} f_0(z) = \frac{z f_1(z) + f_0(0)}{1 + \overline{f_0(0)}\, z f_1(z)}; \;\cdots f_n(z) = \frac{z f_{n+1} + f_n(0)}{1 + \overline{f_n(0)}\, z f_{n+1}(z)}; \;\cdots .$$

The above process is known as Schur's algorithm, which is intimately related with the Gram–Schmidt orthogonalization algorithm. This nontrivial fact was first discovered by Geronimus in 1943 [15].

Theorem (Geronimus [15]). *Let $\sigma \in \mathscr{P}$ and f be the Schur function of σ. Then*

(1.4)

$$a_n = f_n(0) = -\frac{\overline{\varphi_{n+1}(0)}}{k_{n+1}}, \qquad n = 0, 1, \ldots .$$

DEFINITION. The numbers $\{a_n\}_{n\geqslant 0}$ in (1.4) are called the *Geronimus parameters* of σ. The functions $f_n = f_n^\sigma$ are called the *Schur iterates* for σ.

The main idea of the proof of Geronimus' theorem is to apply Tchebysheff's classical approach, based on the theory of continued fractions, in the setting of orthogonal polynomials on the unit circle. It is the theory of continued fractions which explains the relationship between these two important algorithms.

The polynomials $\{\varphi_n\}_{n\geqslant 0}$ are uniquely determined by the *Geronimus parameters*

$$a_n = -\frac{\overline{\varphi_{n+1}(0)}}{k_{n+1}}, \qquad n = 0, 1, \ldots .$$

This fact follows from the recurrence formulae [46, (11.4.7), p. 293; 17, (1.2-2')]

$$\begin{aligned} k_n\varphi_{n+1} &= k_{n+1}z\varphi_n + \varphi_{n+1}(0)\,\varphi_n^* \\ k_n\varphi_{n+1}^* &= k_{n+1}\varphi_n^* + \overline{\varphi_{n+1}(0)}\,z\varphi_n, \end{aligned} \tag{1.5}$$

which can be considered as the Euler–Wallis recurrence for a special continued fraction, associated with F_σ. Here

$$p_n^*(z) = z^n\overline{p_n\left(\frac{1}{\bar{z}}\right)}$$

is the *reversed* $*$-polynomial of a polynomial p_n of degree n in z.

In [15, 16] Geronimus obtained a beautiful formula for the development of F_σ into a continued fraction

$$F_\sigma = \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\,d\sigma(\zeta) = 1 + \frac{2a_0z}{1-a_0z} - \frac{(1-|a_0|^2)\dfrac{a_1}{a_0}z}{1+\dfrac{a_1}{a_0}z} - \frac{(1-|a_1|^2)\dfrac{a_2}{a_1}z}{1+\dfrac{a_2}{a_1}z} - \cdots \tag{1.6}$$

and proved that the nth convergent of the continued fraction at the right-hand side of (1.6) is ψ_n^*/φ_n^*. Here $\{\psi_n\}_{n\geqslant 0}$ is the sequence of the *second kind* polynomials, i.e., the polynomials associated with the parameters $\{-a_n\}_{n\geqslant 0}$.

In the modern literature the numbers $-\bar{a}_n$ are called the *reflection coefficients* of σ. Reflection coefficients are used when methods of scattering theory are applied. To the contrary, Geronimus parameters are related with more general methods of the theory of *continued fractions*.

The matrix of the isometry $U(\sigma) f = zf$, $f \in L^2(d\sigma)$, in the basis $\{\varphi_n\}_{n \geqslant 0}$ is called the *Hessenberg matrix*. The Hessenberg matrix has a nice representation in terms of the reflection coefficients; see, for example, [18, p. 401]. Subtracting 1 from both sides of (1.6) and dividing by z, we obtain the development of the resolvent of $U(\sigma)$ (= the Hessenberg matrix) into a continued fraction. Substituting (1.1) in (1.6) and solving the equation obtained with respect to f, we get:

(1.7)
$$f(z) = \frac{a_0}{1} - \frac{(1-|a_0|^2)\dfrac{a_1}{a_0}z}{1+\dfrac{a_1}{a_0}z} - \frac{(1-|a_1|^2)\dfrac{a_2}{a_1}z}{1+\dfrac{a_2}{a_1}z} - \cdots - \frac{(1-|a_{n-1}|^2)\dfrac{a_n}{a_{n-1}}z}{1+\dfrac{a_n}{a_{n-1}}z} - \cdots.$$

One can easily check (see the corresponding formulae in [25, Chap. 2]) that (1.7) is the even part of the Wall continued fraction [48]:

(1.8)
$$f(z) = a_0 + \frac{(1-|a_0|^2)z}{\bar{a}_0 z} \, \frac{1}{+\, a_1 +} \, \frac{(1-|a_1|^2)z}{\bar{a}_1 z} \, \frac{1}{+\cdots+\, a_n +} \, \frac{(1-|a_n|^2)}{\bar{a}_n z} \, {+\cdots}.$$

It is shown in [48] that the algorithm of the Wall continued fraction is nothing but Schur's algorithm (see [29] and [3] for further references).

Remark. There is an opinion that the representation of convergents of (1.6) as quotients of orthogonal polynomials, as well as Geronimus' theorem [15], were already known to Schur. However, it looks like this opinion contradicts real facts. Schur published his well-known papers [43] and [44] in 1917 and 1918. The fact that Schur's algorithm coincides with the algorithm of continued fraction (1.8) was known to Schur (see remarks in [43] and [44]), but one cannot find any mentioning of orthogonal polynomials in [43] and [44]. The theory of orthogonal polynomials on the unit circle was started in a series of papers by Szegő in 1918 (see, for example, [26, p. 128]). At that time Szegő did not give the recurrence relations (1.5). They appeared only in 1939 in the first edition of his book [46]. It is rather obvious therefore that in 1917–1918 there was no clear understanding that Schur's parameters coincide with the parameters introduced later by Geronimus [15]. Nor was there any idea to apply Tchebysheff's approach to orthogonal polynomials on the unit circle (see, however, [49–53]).

1.2. *The space of Schur's parameters.* By Schwarz's lemma Schur's algorithm runs up to infinity except for finite Blaschke products, i.e., the functions f in $\mathbb{D}$ of the form:

$$f(z) = \gamma \cdot \prod_{k=1}^{n} \frac{\overline{\lambda_k}}{|\lambda_k|} \cdot \frac{\lambda_k - z}{1-\overline{\lambda_k} z}, \qquad \lambda_k \in \mathbb{D}, \quad k = 1, 2, \ldots, n, \quad |\gamma| = 1.$$

To include finite sets of the parameters of finite Blaschke products in our consideration we define a special set $\mathscr{S}^\infty$ of contracting sequences on the set of all nonnegative integers $\mathbb{Z}_+$. Namely, we consider sequences x whose domain $\mathscr{D}(x)$ is either $\mathbb{Z}_+$ or a finite segment $[0, n] \cap \mathbb{Z}_+$, $n = 0, 1, \ldots$. If $\mathscr{D}(x) = \mathbb{Z}_+$, then we say that $x \in \mathscr{S}^\infty$ if $|x_n| < 1$ for all $n \in \mathbb{Z}_+$. If $\mathscr{D}(x) = [0, n] \cap \mathbb{Z}_+$ for some finite n, then we say that $x \in \mathscr{S}^\infty$ if $|x_k| < 1$ for $k = 0, 1, \ldots, n-1$ and $|x_n| = 1$. We consider in $\mathscr{S}^\infty$ the topology of pointwise convergence.

The following theorem obtained in [29, Lemma 4.11 and the remark to it] provides a useful tool for the study of orthogonal polynomials.

THEOREM 1.1. *The mapping* $\mathscr{S} : \mathscr{B} \to \mathscr{S}^\infty$

$$\mathscr{S} f = (a_0, a_1, \ldots), \tag{1.9}$$

where $\{a_n\}_{n \geqslant 0}$ *are the Schur parameters of* f, *is a homeomorphism.*

COROLLARY 1.2. *The topological space* $\mathscr{S}^\infty$ *is a metrizable compactum.*

COROLLARY 1.3. *The mapping* $\mathscr{S} \circ \mathscr{H} : \mathscr{P} \to \mathscr{S}^\infty$ *is a homeomorphism.*

Theorem 1.1 allows one to reduce the study of weak convergence of measures to the study of pointwise convergence of their Schur parameters.

Let $q_0 + K_1^\infty(p_n/q_n)$ be any continued fraction with convergents P_n/Q_n. It is well known that

$$\begin{aligned} \frac{P_n}{P_{n-1}} &= q_n + \frac{p_n}{q_{n-1}} \, {}_{+} \, \frac{p_{n-1}}{q_{n-2}} \, {}_{+ \cdots +} \, \frac{p_1}{q_0}, \\ \frac{Q_n}{Q_{n-1}} &= q_n + \frac{p_n}{q_{n-1}} \, {}_{+} \, \frac{p_{n-1}}{q_{n-2}} \, {}_{+ \cdots +} \, \frac{p_2}{q_1}. \end{aligned} \tag{1.10}$$

One can easily obtain (1.10) iterating the Euler–Wallis formulate

$$\begin{gathered} P_n = q_n P_{n-1} + p_n P_{n-2}, \\ Q_n = q_n Q_{n-1} + p_n Q_{n-2}, \\ n = 1, 2, \ldots, \end{gathered}$$

which ara valid for the numerators P_n and the denominators Q_n of the nth convergent of the continued fraction.

The important formulae (1.10) explains why finite "reversed" continued fractions are so useful in convergence theory. This observation was ingeniously applied by Galois [12] to the developments of roots of quadratic equations into regular continued fractions. It turns out that a similar phenomenon takes place for orthogonal polynomials.

Taking the quotient of formulae (1.5), we obtain that

$$b_n = \frac{zb_{n-1} - \bar{a}_{n-1}}{1 - za_{n-1}b_{n-1}}, \ldots, b_0 = 1, \tag{1.11}$$

where $b_n \stackrel{\text{def}}{=} \varphi_n/\varphi_n^*$ is a finite Blaschke product, since the zeros of φ_n are located in $\mathbb{D}$ [46, Theorem 11.4.1] (see also (2.26) below). It follows that

$$\mathscr{S} b_n = (-\bar{a}_{n-1}, -\bar{a}_{n-2}, \ldots, 1). \tag{1.12}$$

Due to this reason we give the following definition.

DEFINITION. If $\sigma \in \mathscr{P}$, then the quotient

$$b_n = \frac{\varphi_n}{\varphi_n^*}$$

is called the *nth inverse Schur iterate* for σ.

Our present research is based on one of the main results of [29].

THEOREM 1.4 [29, Theorem 3]. *Let $\sigma \in \mathscr{P}$, $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$, $\{f_n\}_{n\geqslant 0}$ the Schur iterates for σ, and $\{b_n\}_{n\geqslant 0}$ the inverse Schur iterates for σ. Then*

$$\mathscr{H}(|\varphi_n|^2\, d\sigma) = f_n b_n, \qquad n = 0, 1, \ldots. \tag{1.13}$$

Although it is not so easy to control the Schur parameters of $f_n b_n$, the fact that $\mathscr{S} f_n$ and $\mathscr{S} b_n$ are known leads to some conclusions on the behavior of $|\varphi_n|^2\, d\sigma$.

1.3. *Weak asymptotic formulae.* The notion of weak asymptotic formulae for orthogonal polynomials goes back to the classical result by Szegő [46, Theorem 12.1.1],

$$\varphi_n^*(z) \rightrightarrows D_\sigma^{-1}(z) = \exp\left\{-\frac{1}{2}\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \log\sigma'\, dm\right\}, \tag{1.14}$$

uniformly on compact subsets of $\mathbb{D}$. For a probability measure σ the quantity on the right-hand side of (1.14) does not vanish if and only if σ is a Szegő measure:

$$\int_{\mathbb{T}} \log\sigma'\, dm > -\infty. \tag{1.15}$$

Here m, is the normalized ($m(\mathbb{T})=1$) Lebesgue measure on $\mathbb{T}$ and σ' is the Lebesgue derivative $d\sigma/dm$ of σ. If we put $z=0$ in (1.14), then we obtain that for Szegő measures the increasing sequence

$$k_n = \left\{\prod_{j=0}^{n-1} (1-|a_j|^2)\right\}^{-1/2} \tag{1.16}$$

is bounded. Moreover, Geronimus [15] proved that σ is a Szegő measure if the sequence in (1.16) is bounded. It follows that (1.14) is in fact equivalent to the corresponding asymptotic formula for the *monic* orthogonal polynomials

$$\Phi_n(z) \stackrel{\text{def}}{=} k_n^{-1}\varphi_n(z), \tag{1.17}$$

which in turn implies that

$$\frac{\Phi^*_{n+1}}{\Phi^*_n} \rightrightarrows 1 \tag{1.18}$$

uniformly on compact subsets of $\mathbb{D}$.

The description of σ satisfying (1.18) was the subject of quite a long study which resulted in the following solution.

We say that $\sigma \in \mathscr{P}$ is a *Rakhmanov measure* (notationally $\sigma \in \mathscr{R}$) if

$$(*)-\lim_n |\varphi_n|^2\, d\sigma = dm \tag{1.19}$$

in the $(*)$-weak topology of $M(\mathbb{T})$.

THEOREM 1.5 [29, Theorems 4 and 7.4]. *The following conditions are equivalent.*

(1) *A probability measure σ is a Rakhmanov measure;*

(2) *The Geronimus parameters $\{a_n\}_{n\geqslant 0}$ of σ satisfy the Máte–Nevai condition*

$$\lim_n a_{n+k}a_n = 0, \qquad k=1,2,\ldots; \tag{1.20}$$

(3) *Ratio asymptotic formula* (1.18) *holds for σ.*

Theorem 1.5 leads to the natural problem of describing σ in $\mathscr{P}$ such that the limit

$$(*)-\lim_n |\varphi_n|^2\, d\sigma$$

exists. It also shows that this problem is related with the description of measures σ for which the ratio

$$\frac{\Phi^*_{n+1}}{\Phi^*_n}$$

converges uniformly on compact subsets of $\mathbb{D}$. In the present paper we give a complete solution to both problems.

1.4. *Ratio asymptotic measures.*

DEFINITION. A probability measure σ is called ratio asymptotic if the following limit

$$G_\sigma(z) := \lim_n \frac{\Phi^*_{n+1}(z)}{\Phi^*_n(z)} \tag{1.21}$$

exists for every z in $\mathbb{D}$.

Our first purpose is to describe all possible functions G_σ on the left-hand side of (1.21) and to classify ratio asymptotic measures.

DEFINITION. A probability measure σ is said to satisfy the López condition if there exist $a \in (0, 1]$ and $\lambda \in \mathbb{T}$ such that

$$\lim_n |a_n| = a, \qquad \lim_n \frac{a_{n+1}}{a_n} = \lambda. \tag{L}$$

Recall that a continued fraction $q_0 + K_1^\infty(p_n/q_n)$ is called *limit periodic* (or also 1-*term limit periodic*) [38, Band II, Sect. 19, p. 92] if both limits $\lim_n p_n$ and $\lim_n q_n$ exist.

LEMMA 1.6. *Let $a_n \neq 0$ for $n = 0, 1, 2, \ldots$ be a sequence in $\mathbb{D}$ and let any of the continued fractions* (1.6) *or* (1.7) *be* 1-*term limit periodic for some z, $z \neq 0$. Then either $a_n = o(q^n)$ for every $q > 0$, or $\lim_n a_n = 0$ and the limit $\lim_n a_n/a_{n-1} = \lambda$ exists and $0 < |\lambda| \leqslant 1$, or else the probability measure corresponding to the parameters $\{a_n\}_{n \geqslant 0}$ satisfies the López condition.*

Proof. Since $\lim_n (1 + z a_n/a_{n-1})$ exists for some $z \neq 0$, we conclude that

$$\lim_n \frac{a_n}{a_{n-1}} = \lambda,$$

where $\lambda \in \mathbb{C}$. If $\lambda = 0$, then $a_n = o(q^n)$ for every $q > 0$ and the proof is finished. If $\lambda \neq 0$, then since the limit $\lim_n (1 - |a_n|^2)\, z a_n/a_{n-1}$ exists for some

$z \neq 0$, we obtain that $\lim_n(1-|a_n|^2)$ exists too. If $\lim_n|a_n| = a$ and $a = 0$, then $0 < |\lambda| \leqslant 1$. Indeed, otherwise we would have $\lim |a_n| = \infty$, which is impossible. If $a \neq 0$, then $\{a_n\}_{n \geqslant 0}$ satisfies (L). ∎

Clearly, if σ satisfies (L), then both Geronimus continued fractions (1.6) and (1.7) are limit-periodic. This indicates that in the general theory of orthogonal polynomials measures satisfying (L) might be considered as a replacement for measures in Nevai's class. However, the following theorem demonstrates an important difference.

THEOREM A. *A probability measure σ is ratio asymptotic if and only if either σ is a Rakhmanov measure or σ satisfies the López condition* (L).

The study of ratio asymptotic measures on the real line was initiated in [3]. For $a \in (0, 1]$ Theorem A was proved in [2, Theorem 1]. Based on the approach developed in [29], we give below new proofs to some important results of [2].

It is convenient to classify measures satisfying (L) in terms of their Schur functions. Given $a \in (0, 1)$ we consider the Schur function f_a corresponding to the constant parameters $a_n = a$, $n = 0, 1, \ldots$. By (1.3) this function satisfies the quadratic equation $azX^2 + (1-z)X - a = 0$. It follows that

$$f_a(z) = \frac{(z-1) + \sqrt{(1-z)^2 + 4a^2 z}}{2az}, \tag{1.22}$$

where the branch of the square root in (1.22) is chosen to take value 1 at $z = 0$. The algebraic function f_a has two branch points on $\mathbb{T}$

$$z_{1,2} = (1-2a^2) \pm 2ai\sqrt{1-a^2} = \exp(\pm i\alpha), \qquad \sin(\alpha/2) = a, \tag{1.23}$$

which are the endpoints of the arc $\Delta_\alpha = \{\exp(i\theta) \colon \alpha \leqslant \theta \leqslant 2\pi - \alpha\}$. The function f_a is unimodular on $\mathbb{T} \setminus \Delta_\alpha$ and $|f_a(\zeta)| < 1$ at every interior point ζ of Δ_α on $\mathbb{T}$, assuming that $f_a(\zeta) = \lim_{r \to 1-0} f_a(r\zeta)$. The branch f_a of the algebraic function (1.22) is a single-valued analytic function on the cut plane $\mathbb{C} \setminus \Delta_\alpha$.

Since $f_n \in \mathscr{B}$, f_a is the Schur function of some probability measure on $\mathbb{T}$. This measure is the equilibrium measure of the arc Δ_α with respect to the logarithmic kernel.

The construction below provides a simple way to determine the densities of the equilibrium measures for pairs of arcs of $\mathbb{T}$ symmetric with respect to the real axis.

For any $a, a' \in [0, 1]$ we have

$$-1 \leqslant -aa' - \sqrt{(1-a^2)(1-a'^2)} \leqslant -aa' + \sqrt{(1-a^2)(1-a'^2)} \leqslant 1,$$

which implies that the equations

$$\cos\theta_1 = -aa' + \sqrt{\omega_1}, \qquad \cos\theta_2 = -aa' - \sqrt{\omega_1},$$
$$\omega_1 \overset{\text{def}}{=} (1-a^2)(1-a'^2)$$

have the unique solutions θ_1 and θ_2 in $[0, \pi]$. These solutions satisfy $\theta_1 \leqslant \theta_2$, where $\theta_1 = \theta_2$ if and only if $\max(a, a') = 1$. Excluding the special case $\theta_1 = \theta_2$ from our consideration for the time being, we consider the two closed symmetric arcs

$$J(a, a') \overset{\text{def}}{=} \{\exp(i\theta)\colon \theta_1 \leqslant \theta \leqslant \theta_2\} = J,$$
$$\bar{J}(a, a') \overset{\text{def}}{=} \{\exp(-i\theta)\colon \theta_1 \leqslant \theta \leqslant \theta_2\} = \bar{J}$$

on $\mathbb{T}$, which are projected by $\exp(i\theta) \to \cos\theta$ onto the segment $[\cos\theta_2, \cos\theta_1]$. Let us define now the density $p_{a,a'}$ on $\mathbb{T}$ to be identically zero on $\mathbb{T} \setminus (J \cup \bar{J})$ and put

$$p_{a,a'}(e^{i\theta}) = \frac{|\sin\theta|}{\sqrt{(\cos\theta - \cos\theta_2)(\cos\theta_1 - \cos\theta)}}. \tag{1.24}$$

if $\exp(i\theta) \in J \cup \bar{J}$. Following standard notations [46, (11.5.1); 17, (9.3)], we denote by $w_{a,a'}\,dx/\pi$ the image of the measure $p_{a,a'}\,dm$ under the orthogonal projection $\exp(i\theta) \to \cos\theta$. It is clear that $w_{a,a'}$ is supported by $[\cos\theta_2, \cos\theta_1]$ and that

$$w_{a,a'}(x) = \frac{1}{\sqrt{(\cos\theta_1 - x)(x - \cos\theta_2)}}. \tag{1.25}$$

The change of variables formulae

(1.26)
$$\int_{\mathbb{T}} f p_{a,a'}\,dm = \frac{1}{\pi}\int_{\cos\theta_1}^{\cos\theta_2} g(x)\, w_{a,a'}(x)\,dx = \frac{1}{\pi}\int_{-1}^{1} \frac{g(x\sqrt{\omega_1} - aa')}{\sqrt{1-x^2}}\,dx,$$

where $g(\cos\theta) = \frac{1}{2}\{f(e^{i\theta}) + f(e^{-i\theta})\}$ show (put $f \equiv 1$) that $p_{a,a'}\,dm \in \mathscr{P}$.

It is well known (see, e.g., [23]) that

$$\frac{1}{\pi}\int_{-1}^{1} \frac{dx}{(z-x)\sqrt{1-x^2}} = \frac{1}{\sqrt{z^2-1}} = \frac{1}{z\,-}\,\frac{1/2}{z\,-}\,\frac{1/4}{z\,-}\,\frac{1/4}{z\,-}\cdots. \tag{1.27}$$

To prove (1.27) we notice that (1.26) with $a = a' = 0$ transforms the left-hand side of (1.27) to an integral over $\mathbb{T}$, which can be calculated by

Cauchy's theorem. On the other hand, the development of $(z^2-1)^{1/2}$ into a periodic continued fraction can be obtained by iterations of an elementary identity

$$\sqrt{z^2-1} = z - \frac{1}{z+\sqrt{z^2-1}}$$

followed by an equivalence transformation (see Sect. 2, (2.2) for a definition). It follows from (1.26) and (1.27) that

$$\frac{1}{\pi}\int_{-1}^{1} \frac{w_{a,a'}\,dx}{(z-x)} = \frac{1}{z+aa'} \, \frac{}{-} \, \frac{\omega_1/2}{z+aa'} \, \frac{}{-} \, \frac{\omega_1/4}{z+aa'} \, \frac{}{-} \, \frac{\omega_1/4}{z+aa'} \, \frac{}{-} \cdots . \tag{1.28}$$

We define the probability measure $\mu_{a,a'}$ as follows

(1.29)
$$d\mu_{a,a'} = \begin{cases} p_{a,a'}\,dm & \text{if} \quad \max(a,a')<1 \\ \frac{1}{2}(\delta_t+\delta_{\bar t}), & t=\exp(i\,\operatorname{arc}\cos(-aa')), \quad \text{if} \quad \max(a,a')=1. \end{cases}$$

Given a measure μ on $\mathbb{T}$ and $\lambda \in \mathbb{T}$ we denote by μ_λ the measure defined on Borel subsets E of $\mathbb{T}$ by $\mu_\lambda(E)=\mu(\lambda E)$. This construction applied to $\mu_{a,a'}$ gives $\mu_{a,a',\lambda}$. In case $0<a=a'<1$ we denote $\mu_{a,a',\lambda}$ by $\nu_{a,\lambda}$. The probability measure $\nu_{a,\lambda}$ is the equilibrium distribution for $\bar\lambda\Delta_\alpha$.

1.5. *The López conditions and weakly asymptotic measures.* Let us return to Theorem A and compare it with Theorem 1.5. By definition Rakhmanov measures satisfy (1.19). We prove that similar relations hold for measures satisfying the López condition.

THEOREM B. *A probability measure σ on, $\mathbb{T}$ satisfies the López condition* (L) *with parameters* (a,λ) *if and only if*

$$(*)-\lim_n |\varphi_n|^2\,d\sigma = d\nu_{a,\lambda}, \tag{1.30}$$

where $d\nu_{a,\lambda}$ is the equilibrium measure for the arc $\bar\lambda\Delta_\alpha$, $\sin\alpha/2=a$, with respect to the logarithmic kernel, if $0<a<1$, and $d\nu_{1,\lambda}=\delta_{-\bar\lambda}$ if $a=1$.

Remark. The implication (L) $\Rightarrow$ (1.30) was proved by Golinskii [20, Theorem 4] by the method of the Hessenberg matrix. In a recent paper [4] Bello and Miña proved that measures σ supported by $\bar\lambda\Delta_\alpha$ and such that $\sigma'>0$ a.e. on $\bar\lambda\Delta_\alpha$ satisfy

$$(*)-\lim_n |\varphi_n|^2\,\sigma'\,dm = d\nu_{a,\lambda}. \tag{1.31}$$

This shows that the class of López measures in this respect behaves similar to the class of Rakhmanov measures.

Since $z \to \bar{\lambda} z$ is a rotation about $z = 0$, in what follows we often assume that $\lambda = 1$ and also put $d\nu_a \overset{\text{def}}{=} d\nu_{a,1}$.

If $0 < a < 1$, then $\dim L^2(d\nu) = +\infty$. Therefore the sequence $\{\varphi_n\}_{n \geqslant 0}$ of orthogonal polynomials in $L^2(d\nu)$ is infinite. It is natural to expect that $\nu_{a,\lambda}$ satisfies the López condition, so that (1.30) holds with $\sigma = \nu_{a,\lambda}$. Since by (1.29) $\nu'_n > 0$ a.e. on Δ_α this fact follows from a general theorem by López [5]. However, for the completeness of the theory it is important to have explicit formulae for the Geronimus parameters $\{a_n\}_{n \geqslant 0}$ of ν_a. We prove that the a_n's can be obtained from the recurrence

$$a_0 = -a^2,\ a_n = (1 - a^2)^{n+1} \prod_{k=0}^{n-1} (1 - a_k^2)^{-1} - 1, \qquad n = 1, 2, 3, \ldots. \tag{1.32}$$

We also prove that

$$a_n = -a + c \left(\frac{1-a}{1+a} \right)^n (1 + o(1)), \qquad n \to \infty, \tag{1.33}$$

which obviously implies that $\nu_{a,\lambda}$ satisfies (L).

Definition. A probability measure σ is called weakly asymptotic if

$$(*) - \lim_n |\varphi_n|^2 \, d\sigma = d\mu \tag{1.34}$$

for some $\mu \in \mathscr{P}$, $\{\varphi_n\}_{n \geqslant 0}$ being the orthogonal polynomials in $L^2(d\sigma)$.

By Theorems A and B every ratio asymptotic measure is weakly asymptotic. Our next goal is to describe weakly asymptotic measures.

Definition. A measure σ is said to satisfy the López condition (L_2) if there exist $a, a' \in (0, 1]$ and $\lambda \in \mathbb{T}$ such that

(L_2)

$$\lim_n |a_{2n}| = a, \qquad \lim_n |a_{2n+1}| = a', \qquad \lim_n \frac{a_{2n}}{a_{2n-1}} = \frac{a}{a'} \lambda, \qquad \lim_n \frac{a_{2n+1}}{a_{2n}} = \frac{a'}{a} \lambda.$$

For measures $\mu_{a,a',\lambda}$ we prove the following analogue of Theorem B.

Theorem C. *A probability measure σ on $\mathbb{T}$ satisfies the López condition (L_2) with parameters (a, a', λ) if and only if*

$$(*) - \lim_n |\varphi_n|^2 \, d\sigma = d\mu_{a,a',\lambda}.$$

If $\max(a, a') < 1$, *then $\mu_{a,a',\lambda}$ is the equilibrium measure for $\bar{\lambda} J(a, a') \cup \bar{\lambda} \bar{J}$.*

The Geronimus parameters $\{a_n\}_{n\geqslant 0}$ of $\mu_{a,a'}$ satisfy $a_{-1}=0$, $a_0=-aa'$:

$$(1.35)\qquad \begin{aligned} a_{2n} &= \frac{a_{2n-2}(1+a_{2n-1})-2aa'}{1-a_{2n-1}}, \\ a_{2n+1} &= \frac{(1-a^2)(1-a'^2)}{(1-a_{2n}{}^2)(1-a_{2n-1})}-1, \qquad n=1,2,\dots. \end{aligned}$$

Notice that $\mu_{a,a'}=\mu_{a',a}$.

By the definition of $\mu_{a,a'}$ (see (1.24) and(1.27)) this measure is symmetric with respect to the real axis. Therefore the Geronimus parameters of $\mu_{a,a'}$ are real and can be recovered from coefficients of the J-fraction by the following theorem.

THEOREM 1.7 (Geronimus [16, Theorem 31.1]). *Let σ be a probability measure on $\mathbb{T}$ symmetric with respect to the real axis and ψ be the image of σ on $[-1,1]$ under the projection $x=\frac{1}{2}(z+\frac{1}{z})$ of $\mathbb{T}$ onto $[-1,1]$. Then the coefficients of the J-fraction*

$$(1.36)\qquad \int_{-1}^{1}\frac{d\psi(x)}{z-x}=\frac{1}{z-\alpha_1}\,\underset{-}{}\,\frac{\lambda_2}{z-\alpha_2}\,\underset{-\cdots-}{}\,\frac{\lambda_n}{z-\alpha_n}\,\underset{-\cdots}{}$$

are related with the Geronimus parameters $\{a_n\}_{n\geqslant 0}$ of σ by the following formulae

$$(1.37)\qquad \begin{aligned} \alpha_{n+1} &= -\frac{a_{2n-2}(1+a_{2n-1})-a_{2n}(1-a_{2n-1})}{2}, \\ &\qquad n=1,2,\dots,\quad \alpha_1=a_0; \\ \lambda_{n+2} &= \frac{(1-a_{2n-1})(1-a_{2n}^2)(1+a_{2n+1})}{4}, \\ &\qquad n=1,2,\dots,\quad \lambda_2=\frac{(1-a_0^2)(1+a_1)}{2}. \end{aligned}$$

Conversely, if $\{P_n\}_{n\geqslant 0}$ are the monic polynomials orthogonal in $L^2(d\psi)$ and $\{\lambda_n\}_{n\geqslant 2}$, $\{\alpha_n\}_{n\geqslant 0}$ are the coefficients of the J-fraction (1.36), *then the Geronimus parameters $\{a_n\}_{n\geqslant 0}$ of σ are defined by*

$$(1.38)\qquad a_{2n+1}=1-u_{n+1}-v_{n+1},\qquad a_{2n}=\frac{v_n-u_n}{v_n+u_n},$$

where

$$
(1.39)\qquad
\begin{aligned}
u_n &= \frac{P_{n+1}(1)}{P_n(1)} = 1-\alpha_{n+1} - \frac{\lambda_{n+1}}{1-\alpha_n} \,\underset{-}{}\, \frac{\lambda_n}{1-\alpha_{n-1}} \,\underset{-}{}\cdots\underset{-}{}\, \frac{\lambda_2}{1-\alpha_1},\\
v_n &= \frac{P_{n+1}(-1)}{P_n(-1)} = 1+\alpha_{n+1} - \frac{\lambda_{n+1}}{1+\alpha_n} \,\underset{-}{}\, \frac{\lambda_n}{1+\alpha_{n-1}} \,\underset{-}{}\cdots\underset{-}{}\, \frac{\lambda_2}{1+\alpha_1}.
\end{aligned}
$$

We refer the reader to [16, Theorem 31.1] for the proof of this important result. Notice that the right-hand side identities of (1.39) are direct corollaries of (1.10) and of the fundamental fact of the theory of orthogonal polynomials, which says that the denominators of the convergents of (1.36) are the orthogonal polynomials in $L^2(d\psi)$; see, e.g., [46, Sect. 3.5]. In fact, Theorem 1.7 can also be deduced from Szegő's classical theorem [46, Theorem 11.5] by comparing the coefficients of the Szegő polynomials orthogonal in $L^2(d\sigma)$ with the corresponding-coefficients of the P_n's.

Proof of (1.35). Since $\mu_{a,a'}$ is symmetric with respect to the real axis, we can apply Theorem 1.7 to $\sigma=\mu_{a,a'}$. Since $d\psi=\pi^{-1}w_{a,a'}\,dx$, where $w_{a,a'}$ is defined by (1.25), we conclude from (1.28) that

$$\alpha_n=-aa',\quad n=1,2,\ldots;\qquad \lambda_2=\omega_1/2,\qquad \lambda_n=\omega_1/4,\quad n=3,4,\ldots.$$

Hence the first formula of (1.37) is nothing but the first formula of (1.35). The second line in (1.37) yields the second formula (1.35). ∎

The following theorem shows that $\mu_{a,a'}$ satisfies the López condition (L_2) if $a, a'\in(0,1]$. It follows from Theorem C that

$$\operatorname{Lim}(\mu_{a,a',\lambda})=\{\mu_{a,a',\lambda}\}.$$

THEOREM D. *Let* $0\leqslant a\leqslant a'\leqslant 1$ *and let* $\{a_n\}_{n\geqslant 0}$ *be defined by* (1.35). *Then there exists* q, $0<q<1$, *such that*

$$a_{2n}=-a+o(q^n),\quad a_{2n+1}=-a'+o(q^n),\qquad n\to\infty.$$

The following result, completes the description of weakly asymptotic measures. For any integer $k=0,1,\ldots$, the formula

$$(1.40)\qquad \mathscr{H}(\sigma^{(k)})=z^k\mathscr{H}(\sigma)(z^{k+1})$$

determines a transformation $\sigma\to\sigma^{(k)}$ of $\mathscr{P}$. It is easy to see that $(pdm)^{(k)}=p(\zeta^{k+1})\,dm$ for absolutely continuous measures $\sigma=pdm$. For any $\sigma\in\mathscr{P}$ we have

$$\sigma^{(k)}\sim\sum_j\hat\sigma(j)\,\zeta^{j(k+1)},\qquad \hat\sigma(j)=\int_{\mathbb{T}}\bar\zeta^j\,d\sigma,$$

which shows that the transformation $\sigma \to \sigma^{(k)}$ inserts exactly k zero Fourier coefficients between any neighboring Fourier coefficients of σ.

THEOREM E. *A probability measure ν satisfies $Lim(\sigma) = \{\nu\}$ for some $\sigma \in \mathcal{P}$ if and only if there exists $a, a' \in [0, 1]$, $\lambda \in \mathbb{T}$, and an integer $k \geqslant 0$ such that $\nu = \mu^{(k)}_{a, a', \lambda}$. If $\nu = \mu^{(k)}_{a, a', \lambda}$ with $k \geqslant 1$ and $a, a' > 0$, then $Lim_n(\sigma) = \{\nu\}$ if and only if there exists an integer r, $0 \leqslant r \leqslant k$ such that the Geronimus parameters $\{a_n\}_{n \geqslant 0}$ of σ satisfy*

$$\lim_n a_{j+n(k+1)} = 0 \tag{1.41}$$

for any integer j, $0 \leqslant j \leqslant k$, $j \neq r$, whereas the sequence $\{a_{r+n(k+1)}\}_{n \geqslant 0}$ satisfies the Lopez condition (L_2).

1.6. *Markoff measures.* The following Markoff type theorem reveals further generalizations of the notion of a ratio asymptotic measure.

THEOREM F. *Let $\sigma \in \mathcal{P}$ satisfy (L) with parameters (a, λ), $\{\varphi_n\}_{n \geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$, and $\{\psi_n\}_{n \geqslant 0}$ the associated polynomials of the second kind. Then*

$$-\lim_n \frac{\psi_n(z)}{\varphi_n(z)} = \lim_n \frac{\psi_n^*(z)}{\varphi_n^*(z)} = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma(\zeta) \tag{1.42}$$

uniformly on compact subsets of $\mathbb{C} \setminus supp(\sigma)$. The derived set $(supp(\sigma))'$ of $supp(\sigma)$ is the closed arc $\bar{\lambda}\Delta_\alpha$, $\sin \alpha/2 = a$.

The part of Theorem F describing *supp*(σ) was first obtained by Geronimus [14] in the case $\lim_n a_n = a$, $a \in \mathbb{D}$. The proof was based on the paper [47] by Van Vleck on limit-periodic continued fractions. See [18] for the proof using the Hessenberg matrix approach. A more general result was obtained in [2, Theorem 4] for a class of limit-periodic measures of a finite order. In Section 9 we describe how this result can be obtained by the methods of the present paper (see Example 4). In Section 3 we give a proof of Theorem F based on a general result by Perron [38, Band II, Chap. II, Sect. 19, Theorem 2.42] on limit-periodic continued fractions. This proof is analyzed with the purpose of further generalizations. Our analysis leads to a new class Mar$(\mathbb{T})$ of Markoff measures on $\mathbb{B}$, which we are going to discuss now.

For $\sigma \in \mathcal{P}$ we denote by Lim(σ) the derived set of the sequence $\{|\varphi_n|^2\, d\sigma\}_{n \geqslant 0}$ in $\mathcal{P}$. Here φ_n are the orthogonal polynomials in $L^2(d\sigma)$. By Theorem 1.4 $\mathcal{B}(\sigma) \overset{\text{def}}{=} \mathcal{H}(\text{Lim}(\sigma))$ is the set of $\{f_n b_n\}_{n \geqslant 0}$ in $\mathcal{B}$, where f_n and b_n are the Schur iterates for σ.

DEFINITION. A probability measure σ on $\mathbb{T}$ is said to be a Markoff measure, notationally $\sigma \in \operatorname{Mar}(\mathbb{T})$, if $m \notin \operatorname{Lim}(\sigma)$.

Notice that $\operatorname{Mar}(\mathbb{T})$ cannot contain Rakhmanov measures. On the other hand, it is easy to see that any measure σ with $supp(\sigma) \neq \mathbb{T}$ is a Markoff measure. It follows by Theorem 2 that any measure satisfying (L) is a Markoff measure. Since by Rakhmanov's theorem, [29, Corollary 2.3; 40; 41] any Erdős measure σ (i.e., σ with $\sigma' = d\sigma/dm > 0$ a.e. on $\mathbb{T}$) is a Rakhmanov measure, it follows that for any Markoff measure σ its Lebesgue derivative σ' must vanish on a set of positive Lebesgue measures.

Together with $\mathscr{B}(\sigma)$ we consider the derived sets $\mathscr{B}_0(\sigma)$ and $\mathscr{F}_0(\sigma)$ of the sequences

$$\left\{-\frac{a_n}{|a_n|}\, b_{n+1}(z): n = 0, 1, \ldots\right\} \qquad \text{and} \qquad \left\{\frac{\bar{a}_n}{|a_n|}\, f_n(z): n = 0, 1, \ldots\right\}.$$

The following theorem describes Markoff measures in terms of Geronimus parameters.

THEOREM 1.8. *The following conditions are equivalent:*

(1) $\sigma \in Mar(\mathbb{T})$;

(2) $0 \notin \mathscr{B}(\sigma)$;

(3) $0 \notin \mathscr{B}_0(\sigma)$ (*equivalently* $0 \notin \mathscr{F}_0(\sigma)$);

(4) *there exist* $\varepsilon > 0$ *and a positive integer* l *such that the Geronimus parameters* $\{a_n\}_{n \geqslant 0}$ *of* σ *satisfy*

$$\sup_{0 \leqslant j \leqslant l} |a_{n+j}| \geqslant \varepsilon, \qquad n = 0, 1, 2, \ldots. \tag{1.43}$$

Proof. (1) $\Rightarrow$ (2). It follows from the definition, since $\mathscr{H}: \mathscr{P} \to \mathscr{B}$ is a homeomorphism.

(2) $\Rightarrow$ (3). It is obvious, since $\lim_{n \in \Lambda} f_n b_n = 0$ in $\mathbb{D}$ if any of the limits $\lim_{n \in \Lambda} b_n$ or $\lim_{n \in \Lambda} f_n$ are zero in $\mathbb{D}$.

(3) $\Rightarrow$ (4). Suppose to the contrary that (4) does not hold. Then for an arbitrary small $\varepsilon_k > 0$ and an arbitrary large integer l_k we can find an index n_k such that

$$|a_{n_k}| < \varepsilon_k, |a_{n_k+1}| < \varepsilon_k, \ldots, |a_{n_k+l_k}| < \varepsilon_k, \qquad k = 1, 2, \ldots.$$

It follows that we may assume that $\lim_k \varepsilon_k = 0$ and $\lim_k l_k = \infty$. Then the sequence of the elements

$$(-\bar{a}_{n_k+l_k}, -\bar{a}_{n_k+l_k-2}, \ldots, 1)$$

of $\mathscr{S}^\infty$ tends to zero pointwise on $\mathbb{Z}_+$, i.e., in the topology of $\mathscr{S}^\infty$. By (1.10) and Theorem 1.1 the sequence $\{b_{n_k+l_k}\}_{k\geqslant 1}$ tends to zero in $\mathscr{B}$, which obviously contradicts (3). To handle the case $0\notin\mathscr{F}_0$, we consider the sequence $\{a_{n_k+j}\}_{j\geqslant 0}$ of the elements of $\mathscr{S}^\infty$, which tends to zero pointwise. Hence $\lim_k f_{n_k}=0$ in $\mathscr{B}$ and we obtain a contradiction.

(4) $\Rightarrow$ (2). Suppose to the contrary that (2) does not hold. Then there exists an infinite set $\Lambda\subset\mathbb{Z}_+$ such that

$$\lim_{n\in\Lambda} f_n b_n = 0 \tag{1.44}$$

uniformly on compact subsets of $\mathbb{D}$. Consider the family $\{f_n\}_{n\in\Lambda}$ and the set $\mathscr{F}$ of its limit points in $\mathscr{B}$. We claim that $0\notin\mathscr{F}$. Indeed, otherwise we could find an infinite subset Λ_0 of Λ such that $\lim_{n\in\Lambda_0} f_n=0$ in $\mathscr{B}$, which obviously contradicts (1.43) by Theorem 1.1. Hence, there exists an infinite subset $\Lambda_0\subset\Lambda$ such that $\lim_{n\in\Lambda_n} f_n=f\not\equiv 0$. Let r be the radius of any circle $\{z\colon |z|=r\}$ on which f does not vanish. Then by the maximum modulus theorem (1.44) implies that $\lim_{n\in\Lambda_0} b_n=0$ uniformly on $\{z\colon |z|\leqslant r\}$ and therefore in $\mathscr{B}$. This, however, contradicts (4) by Theorem 1.1; see (1.10). ▌

The following corollary follows immediately from Theorem 1.8.

Corollary 1.9. *Let $\sigma\in\mathscr{P}$ with $supp(\sigma)\neq\mathbb{T}$. Then the Geronimus parameters $\{a_n\}_{n\geqslant 0}$ or σ satisfy* (1.43) *for some positive ε and l.*

Corollary 1.10. *Let σ be a measure in $\mathscr{P}$ with Geronimus parameters $\{a_n\}_{n\geqslant 0}$ satisfying*

$$\lim_n |a_n| = 1. \tag{1.45}$$

Then $\sigma\in Mar(\mathbb{T})$.

Proof. It is obvious from (4) $\Rightarrow$ (1) of Theorem 1.8. ▌

Notice that by Rakhmanov's lemma [41, Lemma 4; 29, Corollary 9.5] any measure satisfying (1.45) is singular. Next, if σ satisfies (1.45), then

$$\operatorname{Lim}(\sigma)=\{\delta_t\colon t\in(supp(\sigma))'\};$$

see [29, Theorem 10; 21, Theorem 5]. This, by the way, implies the existence of Markoff measures σ with $supp(\sigma)=\mathbb{T}$.

1.7. *Zeros of the orthogonal polynomials for Markoff's measures.* For an orthogonal polynomial φ_n we denote by $\lambda_{1n},\lambda_{2n},\ldots,\lambda_{nn}$ the zeros of φ_n:

$$\varphi_n(z)=k_n(z-\lambda_{1n})(z-\lambda_{2n})\cdots(z-\lambda_{nn}).$$

It is clear that φ_n and φ_n/φ_n^* have the same zeros. If b is a Blaschke product with zeros $\{\lambda_j\}$, then

$$c(b) \stackrel{\text{def}}{=} \sum_j (1-|\lambda_j|) < \infty,$$

see [13, Chap. II, Sect. 2].

THEOREM G. *Let $\sigma \in Mar(\mathbb{T})$, $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Then*

(1) $\sup_n c(b_n) < \infty$;

(2) *for every r, $0<r<1$, the polynomial φ_n has at most $n(r)$ zeros in $\{z: |z| \leqslant r\}$,*

$$n(r) < \frac{C(\sigma)}{1-r},$$

where $C(\sigma)$ is a positive constant depending only on σ;

(3) *for any open disk Δ centered at $t \in supp(\sigma)$ there exists a positive integer $N(\Delta)$ such that for every $n \geqslant N(\Delta)$ the polynomial φ_n has at least one zero in Δ;*

(4) *for any $t \in \mathbb{T}\backslash supp(\sigma)$ there exists an open disk Δ centered at t and a positive integer $N(t)$ such that for every $n \geqslant N(t)$ the polynomial φ_n does not vanish on Δ.*

COROLLARY 1.11. *Let $\sigma \in Mar(\mathbb{T})$ and $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Then for every $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that for $n > N(\varepsilon)$ every zero of φ_n is in the ε-neighborhood of $supp(\sigma)$ except for a finite number of zeros not exceeding $N(\Delta)$.*

Proof of Corollary 1.11. Let V_ε be the open ε-neighborhood of $supp(\sigma)$. Since $\mathbb{T}\backslash V_\varepsilon$ is compact, we can find a finite family $\Delta_1, ..., \Delta_k$ of disks satisfying (4) of Theorem F and such that

$$\mathbb{T}\backslash V_\varepsilon \subset \bigcup_{j=1}^{k} \Delta_j.$$

Let us choose now the radius r so close to 1 that every point z of $\mathbb{D}$ is either in V_ε or in some Δ_j, $j=1, ..., k$ or else $|z|<r$. By Theorem F (2) the total number of zeros of φ_n in $\{z: |z|<r\}$ is uniformly bounded. By Theorem F

(4) there exists a number N such that for $n > N$ the polynomial φ_n does not vanish on

$$\bigcup_{j=1}^{k} \Delta_j.$$

It follows that any zero of φ_n is either in V_ε or in the disk $\{z\colon |z| < r\}$. ▮

Remark. If $supp(\sigma) \neq \mathbb{T}$, then the polynomial convex hull of $supp(\sigma)$ is $supp(\sigma)$. In this case Corollary 1.11 follows from a general result of Widom (see [42, Theorem 2.4] and [45, Theorem 2.1.1]). The same is true with respect to (1) of Theorem G.

Our first generalization of Theorem G deals with measures satisfying $supp(\sigma) \neq \mathbb{T}$. We denote by $conv_o(\sigma)$ the convex hull of the derived set $(supp(\sigma))'$ and by $supp_p(\sigma) = \{t \in \mathbb{T} : \sigma(t) > 0\}$ the point spectrum of σ. In Theorem 9.2 we prove that the rational functions ψ_n/φ_n converge to $-F_\sigma$ uniformly on compact subsets of

$$\mathbb{C} \setminus (conv_o(\sigma) \cup supp_p(\sigma)).$$

By Theorem 1.8 we have $0 \notin \mathscr{B}(\sigma)$ if and only if $0 \notin \mathscr{B}_0(\sigma)$. In view of (1.8) and (1.10) the set $\mathscr{B}_0(\sigma)$ is responsible for the convergence control of ψ_n/φ_n. In Lemma 9.5 we show that for $\sigma \in \mathrm{Mar}(\mathbb{T})$ the set $Z(\sigma)$ of all zeros of the functions in $\mathscr{B}_0(\sigma)$ is closed in $\mathbb{D}$. In Theorem 9.6 we prove (for σ satisfying $supp(\sigma) \neq \mathbb{T}$) that

$$\frac{\psi_n}{\varphi_n} \rightrightarrows -F_\sigma$$

uniformly on compact subsets of $\mathbb{C} \setminus (supp(\sigma) \cup Z(\sigma))$. It is easy to see that $Z(\sigma) = \varnothing$ if σ satisfies (L). The set $Z(\sigma)$ is finite if $B_0(\sigma)$ is finite, which is the case for example, for σ with limit-periodic (of finite order) Geronimus parameters.

1.8. *Quadrature formulae and two-point Padé approximation.* In Example 3 of Section 9 we construct a measure σ with $supp(\sigma) \neq \mathbb{T}$ such that $Z(\sigma) = (-1, -\delta]$, $\delta > 0$. Examples of such measures are interesting for the problem of two-point Padé, approximation at $z = 0$ and at $z = \infty$. Assuming for simplicity that all zeros $\lambda_{1n}, \lambda_{2n}, \ldots, \lambda_{nn}$ are simple, we may write

$$\frac{\psi_n(z)}{\varphi_n(z)} = \sum_{k=1}^{n} \frac{\psi_n(\lambda_{kn})}{2\lambda_{kn}\varphi_n'(\lambda_{kn})} \frac{z + \lambda_{kn}}{z - \lambda_{kn}}. \tag{1.46}$$

The Christoffel–Darboux coefficients in (1.46)

$$\frac{\psi_n(\lambda_{kn})}{2\lambda_{kn}\varphi'_n(\lambda_{kn})} = \frac{\lambda_{kn}^{n-1}}{b'_n(\lambda_{kn})\,\varphi'_n(\lambda_{kn})^2}, \qquad k = 1, 2, \ldots, n$$

are not positive, as in the case of Markoff's classical theorem on the segment of the real line [46, Theorem 3.5.4], which creates difficulties in applying compactness arguments. Our example shows that two-point Padé approximants to a function holomorphic outside a proper subarc of a unit circle may diverge on a set $(-1, -\delta)$ with $\delta > 0$.

Recall that the fact that ψ_n/φ_n is a two-point Padé approximant of order n at $z = 0$ and of order $n-1$ at $z = \infty$ is equivalent to the validity of the quadrature formula

$$\sum_{k=1}^{n} \frac{\psi_n(\lambda_{kn})}{2\lambda_{kn}\varphi'_n(\lambda_{kn})}\,\Omega(\lambda_{kn}) = \int_{\mathbb{T}} \Omega\, d\sigma \tag{1.47}$$

for any Laurent polynomial of the form

$$\Omega(z) = \sum_{s=-n+1}^{n} \gamma_s z^s.$$

1.9. *A universal classification.* By Theorem E every measure $\nu \in \mathscr{P}$ with $\operatorname{Lim}(\nu) = \{\nu\}$ is the equilibrium measure of a family of closed arcs on $\mathbb{T}$. In case the family of arcs reduces to $J = \mathbb{T}$, we obtain that $\nu = m$, which is the attracting point (in the sense of (1.19)) for Rakhmanov measures.

The orthogonal polynomials in $L^2(d\nu_{a,\lambda})$, where $d\nu_{a,\lambda}$ is the equilibrium measure for the closed arc $\bar{\lambda}\Delta_\alpha$, $0 < \alpha < \pi$, $a = \sin(\alpha/2)$, as well as the orthogonal polynomials in $L^2(t\, d\nu_{a,\lambda})$, t being a continuous and not vanishing function on $\bar{\lambda}\Delta_\alpha$, were studied by Akhiezer in his classical paper [1] (see [19] for a careful presentation and further developments).

The third important case is obtained if we remove from $\bar{\lambda}\Delta_\alpha$ a concentric open subarc. The corresponding equilibrium measure, being symmetric with respect to an axis, can be studied via the standard formulae [17, Chap. IX, (9.3)], which transform the measure $\mu_{a,a',\lambda}$ to the equilibrium measure of the corresponding subsegment of $[-1, 1]$. This trick, however, does not work for general measures σ satisfying the López condition (L_2) with parameters (a, a', λ), for which the measure $\mu_{a,a',\lambda}$ is an attracting point in the sense of Theorem C.

Finally Theorem E says that to cover all possible cases we should take the inverse images of the arcs considered in the second and in the third cases above under the endomorphisms $z \to z^{k+1}$, $k = 1, 2, \ldots$, which keep the

Lebesgue measure m invariant. It is clear that the equilibrium measures ν obtained are exactly the solutions of the equation

$$\operatorname{Lim}(\nu) = \{\nu\}.$$

If we apply the endomorphism $z \to z^2$ to Δ_α, then we obtain two arcs on $\mathbb{T}$ symmetric with respect to the coordinate axis. The corresponding equilibrium measure is $\mu_{0,a}$. By Theorem E we obtain that Theorem C is valid (with obvious modifications) if one of the parameters a, a' is zero.

By Theorem A and Theorem E a ratio asymptotic measure (or more generally a weakly asymptotic measure) is either a Rakhmanov measure or a Markoff measure (see Theorem 1.8). Other important subclasses of Markoff measures are given by measures with asymptotically periodic (of finite order) Geronimus parameters. It is clear that for such measures σ both $\operatorname{Lim}(\sigma)$ and $\mathscr{B}_0(\sigma)$ are finite. The measures with asymptotically periodic (of finite order) Geronimus parameters were the subject of a thorough investigation in [32–36]. In particular, in [36] a description of the set $\operatorname{Lim}(\sigma)$ was given. For these measures one can consider a natural extension of the notion of a ratio asymptotic measure. This notion was studied in [2] and in [37].

Another important subclass of Markoff measures has been introduced in [22]. A measure σ is said to belong to OS ($=\sigma$ is an *opposite Szegő measure*) if the Geronimus parameters $\{a_n\}$ of σ satisfy

$$\sum_n (1-|a_n|^2)^{1/2} < \infty.$$

It is shown in [22] that the orthogonal polynomials of any opposite Szegő measure satisfy asymptotic formulae, which, of course, are not of ratio type. The importance of the class OS is related with the fact that it includes measures related with classical "discrete" orthogonal polynomials.

Thus the set $\mathscr{P}$ splits into three disjoint categories of probability measures. The first is the category of all Rakhmanov of measures, i.e., of measures σ satisfying

$$\operatorname{Lim}(\sigma) = \{m\}.$$

The second category is the set of all Markoff measures, i.e., of measures σ with

$$m \notin \operatorname{Lim}(\sigma).$$

Perhaps the most delicate category is the third consisting of measures σ satisfying

$$\{m\} \subsetneq \operatorname{Lim}(\sigma).$$

It is shown in [22] that this category contains the so-called *universal* measures. A measure σ is called *universal* if

$$\mathrm{Lim}(\sigma) = \mathscr{P}.$$

The very existence of universal measures is not obvious. Nonetheless, not only do universal measures exist, but their Geronimus parameters may satisfy

$$\lim_n \frac{|a_0| + |a_1| + \cdots + |a_n|}{n+1} = 0,$$

i.e., may approach very closely to parameters of measures in Nevai's class. On the other hand, it can be shown that there exist measures σ in the third category such that $\mathrm{Lim}(\sigma)$ is finite.

1.10. *The paper's structure.* The main results of the paper are stated in Section I as theorems numbered by capital Latin letters. In Section 2 we study ratio asymptotic measures and prove Theorem A. In Section 3 we study the Geronimus parameters of equilibrium measures and prove formulae (1.33). The first part of Theorem B is also proved in this section. In Section 4 we study weakly asymptotic measures and prove the second part of Theorem B. In Section 5 equilibrium measures on symmetric arcs of $\mathbb{T}$ are studied and proofs to Theorems C and D are given. The study of weakly asymptotic measures is completed in Section 6, where Theorem E is proved. Section 7 is important from the ideological point of view. Here we prove Theorem F using continued fractions. In Section 8 by applying the method of normal families of holomorphic functions we prove Theorem G, which describes the asymptotic behavior of zeros of orthogonal polynomials of Markoff measures. In Section 9 we study the convergence of continued fraction (1.6).

2. RATIO ASYMPTOTIC MEASURES

Proof of Theorem A splits into three lemmas.

LEMMA 2.1. *Let $\sigma \in \mathscr{P}$, $\{a_n\}_{n\geqslant 0}$ be the Geronimus parameters of σ, and $\{b_n\}_{n\geqslant 0}$ the inverse Schur iterates for σ. Suppose that*

$$\lim_n a_n b_n(z) = B(z) \tag{2.1}$$

for every $z \in \mathbb{D}$. *Then the following limits*

(2.2) $$-\lim_n \bar{a}_n a_{n+k} = c_k$$

exist for $k = 1, 2, \ldots$.

Proof. Since the family $\{a_n b_n\}_{n \geqslant 0}$ is normal in $\mathbb{D}$, the function $B(z)$ is holomorphic in $\mathbb{D}$ and the convergence in (2.1) is uniform on compact subsets of $\mathbb{D}$. Putting $z = 0$ in (2.1), we obtain that

(2.3) $$-\lim_n a_{n+1}\bar{a}_n = B(0).$$

Hence $c_1 = B(0)$. It follows from (1.11) that

(2.4) $$b_{n+1}(z)(1 - z a_n b_n(z)) = z b_n(z) - \bar{a}_n.$$

Multiplying both sides of (2.4) by a_{n+1} and taking (2.3) into account, we obtain

$$\lim_n a_{n+1} b_n(z) = \frac{B(z) - B(0)}{z} - b^2(z) \stackrel{\text{def}}{=} B_{(1)}$$

uniformly on compact subsets of $\mathbb{D}$.

Let us assume that

(2.5) $$\lim_n a_{n+k} b_n(z) = B_{(k)}(z)$$

uniformly on compact subsets of $\mathbb{D}$ and that

$$-\lim_n a_{n+j}\bar{a}_n = c_j, \qquad j = 1, 2, \ldots, k.$$

It follows from (2.5) (put $z = 0$) that

$$-\lim_n a_{n+k+1}\bar{a}_n = B_{(k)}(0).$$

Multiplying both sides of (2.4) by a_{n+k+1}, we obtain that

$$\lim_n a_{n+k+1} b_n(z) = \frac{B_{(k)}(z) - B_{(k)}(0)}{z} - B_{(k)} B \stackrel{\text{def}}{=} B_{k+1}(z).$$

Now the proof is completed by induction. ∎

LEMMA 2.2. *Let $\{a_n\}_{n\geqslant 0}$ be a sequence of complex numbers such that the limits*

$$\lim_n a_{n+k}\bar{a}_n = c_k$$

exist for every $k=1, 2, \ldots$. Then either $c_1 = c_2 = \cdots = c_k = \cdots 0$ or $c_k \neq 0$ for every $k=1, 2, \ldots$. If $c_k \neq 0$, $k=1, 2, \ldots$, and $|a_n|<1$, $n=0, 1, \ldots$, then the sequence $\{a_n\}_{n\geqslant 0}$ satisfies the López condition.

Proof. For $k>s$ we have

$$c_k\bar{c}_s = \lim_n (a_{n+k}\bar{a}_n) \lim_n (\bar{a}_{n+s}a_n) = \lim_n |a_n|^2 (a_{n+k}\bar{a}_{n+s}). \tag{2.6}$$

It is clear that

$$\lim_n a_{n+k}\bar{a}_{n+s} = \lim_n a_{n+s+(k-s)}\bar{a}_{n+s} = c_{k-s}.$$

Suppose that $c_j \neq 0$ for some $j \geqslant 1$. Then for any pair (k, s) with $k-s=j$ we obtain by (2.6) that

$$c_k\bar{c}_s = c_j \lim_n |a_n|^2.$$

Since $c_j \neq 0$, this shows that the limit

$$\lim_n |a_n|^2 = a^2 = \frac{c_k\bar{c}_s}{c_j}$$

exists. If $a=0$, then $c_j = \lim_n a_{n+j}\bar{a}_n = 0$, which contradicts our assumption that $c_j \neq 0$. It follows that $a>0$. Now $a_n \in \mathbb{D}$, $n=0, 1, \ldots$, implies that $a \in (0, 1]$. Finally,

$$\lim_n \frac{a_{n+1}}{a_n} = \lim_n \frac{a_{n+1}\bar{a}_n}{|a_n|^2} = \frac{c_1}{a^2} \in \mathbb{T},$$

which proves the lemma. ▮

LEMMA 2.3. *Let σ satisfy* (L) *for some $a \in (0, 1)$ and $\lambda \in \mathbb{T}$. Then* (1.21) *holds uniformly on compact subsets of $\mathbb{D}$, and G_σ in* (1.21) *satisfies*

$$G_\sigma(z) = \tfrac{1}{2}\left\{(1+\lambda z) + \sqrt{(1-\lambda z)^2 + 4a^2 z}\right\}. \tag{2.7}$$

Proof. It follows from the second recurrence formula (1.5) that

$$\frac{\Phi^*_{n+1}(z)}{\Phi^*_n(z)} = 1 - z a_n b_n(z). \tag{2.8}$$

Although the mapping $\mathscr{S}$ is not linear, still by (1.3)

$$\mathscr{S}(cf) = (ca_0, ca_1, \ldots)$$

for any unimodular constant c, $|c| = 1$. Taking this observation into account and applying it to (1.12) with $c = a_n/|a_n|$, we obtain that

$$\mathscr{S}\left(\frac{a_n}{|a_n|} b_n(z)\right) = \left(-\frac{a_n\bar{a}_{n-1}}{|a_n|}, -\frac{a_n\bar{a}_{n-2}}{|a_n|}, \ldots, \frac{a_n}{|a_n|}\right). \tag{2.9}$$

It follows from the López condition that

$$\lim_n \frac{a_n\bar{a}_{n-k}}{|a_n|} = \lim_n \frac{a_n}{a_{n-k}} \lim_n \frac{|a_{n-k}|^2}{|a_n|} = \lambda^k a, \qquad k = 1, 2, \ldots.$$

Hence the right-hand side of (2.9) converges to

$$x = (-\lambda a, -\lambda^2 a^2, -\lambda^3 a^3, \ldots)$$

in $\mathscr{S}^\infty$ if $0 < a < 1$, and to $x = (-\lambda)$, $\mathscr{D}(x) = \{0\}$, if $a = 1$. In the latter case by Theorem 1.1 the sequence $\{a_n b_n(z)\}_{n\geqslant 0}$ converges to $-\lambda$ uniformly on compact subsets of $\mathscr{D}$. Therefore $G_\sigma = 1 + \lambda z$ which coincides with (2.7) for $a = 1$. If $0 < a < 1$, then it is easy to see that $\mathscr{S}(-\lambda f_a(\lambda z)) = x$ and by Theorem 1.1 $G_\sigma(z) = 1 + a\lambda f_a(\lambda z)$ whereas the convergence in (1.21) is uniform in $\mathbb{D}$. ▌

Proof of Theorem A. It follows from (2.8) that $\sigma \in \mathscr{P}$ is ratio asymptotic if and only if (2.1) holds. If (2.1) holds, then the limits (2.2) exist by Lemma 2.1. By Lemma 2.2 either all these limits are zeros or the Geronimus parameters $\{a_n\}_{n\geqslant 0}$ of σ satisfy the López condition. If the first possibility occurs, then by Theorem 1.5 σ is a Rakhmanov measure and is ratio asymptotic with $G_\sigma \equiv 1$. If σ satisfies the López condition, then σ is ratio asymptotic by Lemma 2.3. ▌

COROLLARY 2.4. *Let σ be a probability measure on $\mathbb{T}$. Then*

$$\frac{\Phi^*_{n+1}(z)}{\Phi^*_n(z)} \rightrightarrows \frac{1}{2}\{(1+\lambda z) + \sqrt{(1-\lambda z)^2 + 4\lambda a^2 z}\} \tag{2.10}$$

in $\mathbb{D}$ for some $a \in (0, 1]$, $\lambda \in \mathbb{T}$, if and only if σ satisfies (L) *with these very values of a and λ.*

Proof. If (2,10) holds, then σ is ratio asymptotic. Since in (2.10) $a > 0$, we obtain that $G_\sigma \not\equiv 1$. This implies by Theorem A that σ satisfies the López condition (L). Next, the values a and λ are uniquely determined by a given G_σ. On the other hand, if σ satisfies (L) then the desired conclusion follows by Lemma 2.3. ▌

In fact, the convergence in (2.10) is uniform on compact subsets of $\mathbb{C}\setminus supp(\sigma)$ [2, Theorem 1]; see also Proposition 3.12.

3. THE EQUILIBRIUM MEASURE OF Δ_α

The following theorem computes the limit measure of the sequence $|\varphi_n|^2\, d\sigma$ for any ratio asymptotic measure σ.

THEOREM 3.1. *Let $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$ for a probability measure σ with Geronimus parameters $\{a_n\}_{n\geqslant 0}$. Suppose that $\{a_n\}_{n\geqslant 0}$ satisfies the López condition* (L) *with parameters* (a, λ). *Then the limit*

$$(*)-\lim_n |\varphi_n|^2\, d\sigma = d\nu_{a,\lambda}$$

exists and $\nu_{a,\lambda}$ is the probability measure on $\mathbb{T}$ associated with the Schur function

$$\mathscr{H}(\nu_{a,\lambda}) = -\lambda f_a^2(\lambda z),$$

where f_a is given in (1.21).

Proof. We consider two sequences

$$\frac{\bar{a}_n}{|a_n|} f_{n+1}(z), \qquad \frac{a_n}{|a_n|} b_{n+1}(z), \qquad n = 0, 1, \ldots,$$

in the ball $\mathscr{B}$. Since we assume that $\{a_n\}_{n\geqslant 0}$ satisfies (L), we obtain that

$$\begin{aligned}
&\lim_n \frac{\bar{a}_n}{|a_n|} a_{n+k} = \lim_n |a_n| \lim_n \frac{a_{n+k}}{a_n} = a\lambda^k, \qquad k = 1, 2, \ldots;\\
&\lim_n -\frac{a_n}{|a_n|} \bar{a}_{n-k} = \lim_n |a_n| \lim_n -\overline{\left(\frac{a_{n-k}}{a_n}\right)} = a\lambda^k, \qquad k = 0, 1, \ldots.
\end{aligned} \tag{3.1}$$

Notice that the Schur parameters of the functions considered are given by

$$\begin{aligned}
&\mathscr{S}\left(\frac{\bar{a}_n}{|a_n|} f_{n+1}\right) = \left(\frac{\bar{a}_n}{|a_n|} a_{n+1}, \frac{\bar{a}_n}{|a_n|} a_{n+2}, \ldots\right);\\
&\mathscr{S}\left(\frac{a_n}{|a_n|} b_{n+1}\right) = \left(-\frac{a_n}{|a_n|} \bar{a}_n, -\frac{a_n}{|a_n|} \bar{a}_{n-1}, \ldots\right).
\end{aligned} \tag{3.2}$$

It follows from (3.1) that the sequences at the right-hand side of (3.2) converge in $\mathscr{S}^\infty$ to the sequences of the Schur parameters of $\lambda f_a(\lambda z)$ and of $-\lambda f_a(\lambda z)$ correspondingly. By Theorem 1.1 we obtain that

$$\lim_n \frac{\bar{a}_n}{|a_n|} f_{n+1}(z) = \lambda f_a(\lambda z), \qquad \lim_n \frac{\bar{a}_n}{|a_n|} b_{n+1}(z) = -f_a(\lambda z)$$

uniformly on compact subsets of $\mathbb{D}$. Hence

$$\lim_n f_{n+1}(z)\, b_{n+1}(z) = -\lambda f_a^2(\lambda z),$$

which completes the proof by Theorem 1.4. ∎

Let us study the measures $\nu_{a,\lambda}$ in more detail. The change of variables $t = \zeta\lambda$, $w = z\lambda$ in (1.3) implies that $\nu_{a,\lambda}(E) = \nu_{a,1}(\lambda E)$ for any Borel subset E of $\mathbb{T}$. Hence without loss of generality we may assume $\lambda = 1$ and study only $\nu_a \overset{\text{def}}{=} \nu_{a,1}$. To study the properties of $\nu_{a,\lambda}$ we apply the method of Schur functions. By Theorem 3.1 $-\lambda f_a^2(\lambda z)$ is the Schur function of $\nu_{a,\lambda}$.

LEMMA 3.2. *The function $-zf_a^2(z)$ is a conformal mapping of $\mathbb{D}$ onto the slit domain $\mathbb{D}\setminus[c, 1)$, $c = a^2(1+\sqrt{1-a^2})^{-2}$.*

Proof. Schur's algorithm (1.6) implies that f_a is a root of the quadratic equation

$$f_a(z) = \frac{zf_a(z)+a}{1+azf_a(z)}, \qquad z \in \mathbb{T}, \tag{3.3}$$

which yields by the well-known formula [13, Chap. I, Sect. (1.5)]

$$1-|f_a(z)|^2 = \frac{(1-a^2)(1-|f_a(z)|^2)}{|1+azf_a(z)|^2}, \qquad z \in \mathbb{T}. \tag{3.4}$$

Therefore for $z \in \mathbb{T}$ either $zf_a(z) \in \mathbb{T}$, in particular $-zf_a^2(z) \in \mathbb{T}$, or $zf_a(z)$ is a point of $\mathbb{D}$ and of the circle $\{w\colon |1+aw|^2 = 1-a^2\}$ orthogonal to $\mathbb{T}$. Putting $z = \exp(i\theta)$ in (1.22), we obtain that

$$zf_a(z) = \begin{cases} \exp(i\theta/2)\,\dfrac{\sqrt{a^2-\sin^2\theta/2}+i\sin\theta/2}{a}, & \text{if } |\theta| \leqslant \alpha \\[2ex] i\exp(i\theta/2)\,\dfrac{\sin\theta/2-\sqrt{\sin^2\theta/2-a^2}}{a}, & \text{if } \alpha < |\theta| \leqslant \pi, \end{cases} \tag{3.5}$$

were $\alpha = 2\arcsin(a)$. It follows that $-\exp(i\theta)\, f_a^2(\exp(i\theta))$ moves along $\mathbb{T}$ in the clockwise direction (from -1 to 1) as θ runs the segment $[0, \alpha]$ from

0 to α. The second formula of (3.5) shows that the mapping $\theta \to -\exp(i\theta) \times f_a^2(\exp(i\theta))$ maps the interval (α, π) onto $(c, 1)$ with

$$c = \left(\frac{1-\sqrt{1-a^2}}{a}\right)^2 = \left(\frac{a}{1+\sqrt{1-a^2}}\right)^2.$$

The behavior of the path on the symmetric interval $(\pi, 2\pi)$ is symmetric. First it returns back to 1 along $(c, 1)$ and then it makes a rotation along $\mathbb{T}$ in the lower half-plane to -1. Hence by the argument principle the function $-zf_a^2(z)$ is univalent in $\mathbb{D}$. ▮

Corollary 3.3. *The function* $-zf_a^2(z)$ *is the conformal mapping of* $\hat{\mathbb{C}} \setminus \Delta_\alpha$ *onto* $\hat{\mathbb{C}} \setminus [c, 1/c]$.

Proof. Since $-zf_a^2(z)$ is unimodular on $\mathbb{T} \setminus \Delta_\alpha$, we obtain from Lemma 3.2 by Schwarz's reflection principle that it is a conformal mapping of $\hat{\mathbb{C}} \setminus \Delta_\alpha$ onto $\hat{\mathbb{C}} \setminus [c, 1/c]$. ▮

Corollary 3.4. *Let* $f_a(z)$ *be the function with equal Schur parameters:* $a_0 = a_1 = \cdots = a > 0$. *Then*

(1) $zf_a(z)$ *maps* $\mathbb{D}$ *conformally onto* $\mathbb{D} \cap \{z : |z+1/a| > \sqrt{a^{-2}-1}\}$;
(2) $f_a(z)$ *maps* $\mathbb{D}$ *conformally onto* $\mathbb{D} \cap \{z : |z-1/a| < \sqrt{a^{-2}-1}\}$;
(3) $1+azf_a(z)$ *maps* $\hat{\mathbb{C}} \setminus \Delta_\alpha$ *conformally onto* $\{z : |z| > \sqrt{1-a^2}\}$.

Proof. (1) It follows from (3.5), which says that the image $\mathbb{T}$ under $zf_a(z)$ is the union of two circular arcs.

(2) It follows from (1), since by (3.3) f_a is zf_a followed by the Möbius transform.

(3) It follows from (1) by Schwarz's reflection principle. ▮

The following proposition is known. However, we provide an algebraic proof, which well be used later in a more complicated setting.

Proposition 3.5. *The probability measure* ν_a *associated to* f_a *by* $\mathscr{H}(\nu_a) = -f_a^2$ *is the equilibrium distribution with respect to the logarithmic kernel for* Δ_α, $\alpha = 2 \arcsin a$.

Proof. It follows from (3.4) and (2.7) that on Δ_α

$$|1+azf_a(z)| = |G_\sigma(z)| = \sqrt{1-a^2}$$

(since $|f_a| < 1$ on Δ_α by (3.5)). By Corollary 3.4(3) we obtain that the Green function of the compact Δ_α is given by

$$g(z) = \log|G_\sigma(z)| - \log\sqrt{1-a^2} + o(1), \qquad z \to \infty.$$

On the other hand, potential theory says (see [31, Chap. 5, Sect. 5, Proposition 5.2]) that the logarithmic potential of the equilibrium distribution μ for the compact Δ_α and the Green function g are related in $\mathbb{C}\setminus\Delta_\alpha$ by the formula

$$\int_{\Delta_\alpha} \log\frac{1}{|\zeta - z|}\, d\mu(\zeta) = -\log\sqrt{1-a^2} - g(z) = -\log|G_\sigma(z)|.$$

It follows that

$$-\int_{\Delta_\alpha} \frac{d\mu(\zeta)}{\zeta - z} = \frac{d}{dz}\left\{\int_{\Delta_\alpha} \log(\zeta - z)\, d\mu(\zeta)\right\} = \frac{d}{dz}\log G_\sigma = \frac{G_\sigma'}{G_\sigma}.$$

Hence

$$\int_{\Delta_\alpha} \frac{\zeta+z}{\zeta-z}\, d\mu(\zeta) = 1 + 2z\int_{\Delta_\alpha} \frac{d\mu(\zeta)}{\zeta - z} = 1 - 2z\frac{G_\sigma'}{G_\sigma}. \tag{3.6}$$

Let $w(z) = zf_a(z)$, $\mathscr{D} = (z-1)^2 + 4a^2 z$. We have $G_\sigma = 1 + aw$, where w satisfies the algebraic equation

$$aw^2 + (1-z)\,w - az = 0. \tag{3.7}$$

Differentiating (3.7), we obtain

$$w' = \frac{w+a}{2aw + (1-z)} = \frac{w+a}{\sqrt{\mathscr{D}}}.$$

Hence

$$\int_{\Delta_\alpha} \frac{\zeta+z}{\zeta-z}\, d\mu(\zeta) = 1 - 2az\frac{w+a}{(1+az)\sqrt{\mathscr{D}}} = 1 - \frac{2aw}{\sqrt{\mathscr{D}}} = \frac{1-z}{\sqrt{\mathscr{D}}}.$$

On the other hand

$$\frac{1 - zf_a^2(a)}{1 + zf_a^2(z)} = \frac{az - aw^2}{az + aw^2} = \frac{(1-z)\,w}{2az + (z-1)\,w} = \frac{1-z}{\sqrt{\mathscr{D}}},$$

since by (3.7)

$$2az + (z-1)\,w = 2aw^2 + w(1-z) = (2aw + 1 - z)\,w = \sqrt{\mathscr{D}}\,w.$$

It follows that $-f_a^2$ is the Schur function of the equilibrium distribution for Δ_α. It remains to observe that $-f_a^2(z)$ is the Schur function for ν_a by Theorem 3.1. ▮

Theorem 3.1 combined with Proposition 3.5 obviously proves Theorem B in the necessity part. We deduce the sufficiency part of Theorem B from more general considerations in the next section. Here we study the Schur parameters of the equilibrium measure for Δ_α.

THEOREM 3.6. *The Geronimus parameters* $\{a_n\}_{n\geqslant 0}$ *of* ν_a *satisfy*

$$a_{n+1}=\frac{1-a^2}{1-a_n}-1, \qquad n=0,1,\ldots, \quad a_0=a^2 \tag{3.8}$$

Proof. Since $\nu_a=\mu_{a,a}$, the parameters $\{a_n\}_{n\geqslant 0}$ satisfy (1.35) with $a'=a$. Let us prove (3.8) by induction. Using the second formula (1.35) for $n=0$, we obtain that

$$a_1=\frac{(1-a^2)(1-a^2)}{(1-a_0)(1+a_0)}-1=\frac{1-a^2}{1-a_0}-1, \tag{3.9}$$

since $1+a_0=1-a^2$. Using the first formula (1.35) for $n=1$, we have

$$a_2=\frac{a_0(1+a_1)-2a^2}{1-a_1}=\frac{(a_0-1)(1+a_1)+2(1-a^2)}{1-a_1}-1=\frac{1-a^2}{1-a_1}-1$$

by (3.9). Now, if we prove by induction that $(1+a_{2n})(1-a_{2n-1})=1-a^2$, then we may conclude by the second formula (1.35) that

$$(1+a_{2n+1})(1-a_{2n})=\frac{(1-a^2)(1-a^2)}{(1+a_{2n})(1-a_{2n-1})}=1-a^2.$$

If we know that $(1+a_{2n+1})(1-a_{2n})=1-a^2$, then by the first formula (1.35) we obtain

$$\begin{aligned}(1+a_{2n+2})(1-a_{2n+1})&=a_{2n}(1+a_{2n+1})-2a^2+1-a_{2n+1}\\&=2(1-a^2)-(1-a_{2n})(1+a_{2n+1})=1-a^2,\end{aligned}$$

which obviously completes the induction. ▮

If we rewrite (3.8) as follows

$$1+a_n=\frac{1-a^2}{1-a_{n-1}^2}(1+a_{n-1}),$$

then we obviously obtain (1.32) by iteration. Applying to $1-a_n$ a different iteration process, we may develop $1-a_n$ into a finite continued fraction of Geronimus' type for $z=1$:

$$1-a_n = 2-\frac{1-a^2}{1-a_{n-1}} = 2-\frac{1-a^2}{2}\,\frac{}{-}\,\frac{1-a^2}{2}\,\frac{}{-}\cdots\frac{}{-}\,\frac{1-a^2}{2-(1-a^2)}. \tag{3.10}$$

Since the periodic continued fraction

$$2+\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{a^2-1}{2}\right)$$

converges to the greatest root of the characteristic equation

$$X^2-2X+(1-a^2)=0,$$

i.e., to $1+a$ (see Theorem 2.1), this can be used to show that $\lim_n a_n=-a$. Unfortunately, the right-hand side of (3.10) is only an intermediate convergent

$$1-a_n=\frac{U_n+wU_{n-1}}{V_n+wV_{n-1}}, \qquad w=a^2-1, \quad n=1,2,$$

of our periodic continued fraction. Therefore we cannot claim that $\lim_n a_n=-a$ right now. To do this we may either apply the theory of periodic continued fraction [25, Theorem 3.2.1] or use the following elementary arguments. By the Euler–Wallis formula (see e.g. [29, (1.2)]) we have

$$V_n=2V_{n-1}-(1-a^2)\,V_{n-2}, \qquad n=1,2,\ldots, \quad V_{-1}=0, \quad V_0=1.$$

It follows that $V_n-V_{n-1}=V_{n-1}-V_{n-2}+a^2V_{n-2}$. By induction we conclude that $\{V_n\}_{n\geqslant 0}$ is an increasing sequence satisfying $V_n-V_{n-1}>a^2$, $n=2,3,\ldots$. It follows from the Euler–Wallis formula that

$$\frac{V_n}{V_{n-1}}=2-(1-a^2)\frac{V_{n-2}}{V_{n-1}}\geqslant 2-(1-a^2)=1+a^2,$$

which shows that $\{V_n\}_{n\geqslant 0}$ increases exponentially fast. Finally, since the convergents U_n/V approach the value of a periodic continued fraction with exponential rate [27, Chap. I, Sect. 7], the identity

$$\frac{U_n+wU_{n-1}}{V_n+wV_{n-1}}-\frac{U_n}{V_n}=\frac{(1-a^2)^n}{V_nV_{n-1}(V_n/V_{n-1}-1+a^2)} \tag{3.11}$$

yields an exponential rate of the approximation of $-a$ by a_n.

However, (1.33) can be proved quite elementary. We need the following elementary lemma.

LEMMA 3.7. *Let A and B be two positive numbers such that the equation, $X^2 - AX + B = 0$ has two different real roots $r < r'$. Let a sequence $\{X_n\}_{n \geqslant 0}$ be defined by*

$$X_{n+1} = A - \frac{B}{X_n}, \qquad n = 0, 1, 2, \ldots, \tag{3.12}$$

with X_0 satisfying $X_0 \in (r, r')$. Then $\{X_n\}_{n \geqslant 0}$ is an increasing sequence in (r, r') such that

$$X_n = r' - \left(\frac{r}{r'}\right)^n c(1 + o(1)), \qquad n \to +\infty, \tag{3.13}$$

for some positive constant c.

Proof. By Vieta's theorem the mapping $\tau(w) = A - B/w$ satisfies $\tau(r) = r$, $\tau(r') = r'$. Since $\tau'(x) = Bx^{-2} > 0$ on (r, r'), we conclude that τ maps the interval onto itself. It follows that $X_n \in (r, r')$, $n = 1, 2, \ldots$. Since by (3.12) the inequality $X_{n+1} > X_n$ is equivalent to the inclusion $X_n \in (r, r')$, we obtain that $\{X_n\}_{n \geqslant 0}$ is an increasing sequence. Passing to the limit in (3.12), we conclude that $\lim_n X_n = r'$.

Let $s_n = r' - X_n$. Then by (3.12) we have

$$(r' - \varepsilon_{n+1})(r' - \varepsilon_n) = A(r' - \varepsilon) - B.$$

Since r' is a root of the equation $X^2 - AX + B = 0$, this implies that

$$\frac{\varepsilon_{n+1}}{\varepsilon_n} = \frac{r}{r' - \varepsilon_n}, \qquad n = 0, 1, \ldots.$$

Therefore $\lim_n \varepsilon_{n+1}\varepsilon_n^{-1} = r/r' < 1$. It follows that $\varepsilon_n = o(q^n)$, $n \to +\infty$, for every $q > r/r'$. Hence the infinite product

$$\varepsilon_0 \prod_{k=0}^{\infty} \left(1 - \frac{\varepsilon_k}{r'}\right)^{-1}$$

converges to a finite value c, $c > 0$. Now the identity

$$\varepsilon_n = \frac{\varepsilon_n}{\varepsilon_{n-1}} \frac{\varepsilon_{n-1}}{\varepsilon_{n-2}} \varepsilon_{n-2} \cdots \frac{\varepsilon_1}{\varepsilon_0} \varepsilon_0 = \left(\frac{r}{r'}\right)^n \varepsilon_0 \prod_{k=0}^{n-1} \left(1 - \frac{\varepsilon_k}{r'}\right)^{-1}$$

obviously yields (3.13). ∎

Proof of 1.33. It is that the equation $X^2-2X+(1-a^2)=0$ has two different real roots $r=1-a<r'=1+a$. By Corollary 3.2 the sequence $X_n=1-a_n$ satisfies (3.12) with $A=2$ and $B=1-a^2$. Since $0<a<1$, we have $X_0=1+a^2\in(r,r')$. Now (1.33) follows from (3.13). ▮

4. WEAKLY ASYMPTOTIC MEASURES

Our main tool in the study of weakly asymptotic measures is the following simple lemma.

LEMMA 4.1. *Let $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$ for $\sigma\in\mathscr{P}$, and f_n, b_n the nth Schur iterate and the nth inverse Schur iterate for σ. Then the limit*

$$(*)-\lim_n |\varphi_n|^2\,d\sigma=d\nu \tag{4.1}$$

exists if and only if

$$\lim_n f_n(z)\,b_n(z)=g(z) \tag{4.2}$$

uniformly on compact subsets of $\mathbb{D}$, where $g\in\mathscr{H}(\nu)$.

Proof. It follows from Theorem 1.4 ▮

First let us describe the probability measures σ for which the limit measure ν in (4.1) exists and satisfies $\mathscr{H}(\nu)(0)=g(0)\neq 0$. Since the Schur function $-\lambda f_a^2(\lambda z)$ of $\nu_{a,\lambda}$ obviously does not vanish at 0, this will allow us to complete the proof of Theorem B.

The following lemma says that (4.2) and the condition $g(0)\neq 0$ imply that the functions

$$\frac{\bar a_n}{|a_n|}zf_{n+1(z)},\qquad \frac{a_n}{|a_n|}b_{n+1} \tag{4.3}$$

are asymptotic solutions to the following quadratic equation

$$X^2+|a_n|\,(1+zg)\,X+zg=0. \tag{4.4}$$

LEMMA 4.2. *Let σ be a probability measure on $\mathbb{T}$ satisfying* (4.1) *with such a measure ν that its Schur function g does not vanish at $z=0$. Let $\{a_n\}_{n\geqslant 0}$ be the Geronimus parameters of σ. Then*

(4.5) $$\left(\frac{\bar{a}_n}{|a_n|} zf_{n+1}\right)^2 + |a_n|\,(1+zg)\left(\frac{\bar{a}_n}{|a_n|} zf_{n+1}\right) + zg = o(1),$$

(4.6) $$\left(\frac{a_n}{|a_n|} b_{n+1}\right)^2 + |a_n|\,(1+zg)\left(\frac{a_n}{|a_n|} b_{n+1}\right) + zg = o(1),$$

uniformly on compact subsets of $\mathbb{D}$ *as* $n \to +\infty$.

Proof. By (1.3) and (1.11) we obtain that

$$zf_n(z)\, b_n(z) = \frac{b_{n+1} + \bar{a}_n}{1 + a_n b_{n+1}} \; \frac{zf_{n+1} + a_n}{1 + \bar{a}_n z f_{n+1}(z)},$$

which is equivalent to the identity

(4.7) $$\begin{aligned} &zf_n b_n (1 + \bar{a}_n z f_{n+1} + a_n b_{n+1} + z\,|a_n|^2\, b_{n+1} f_{n+1}) \\ &\quad = z f_{n+1} b_{n+1} + a_n b_{n+1} + |a_n|^2 + \bar{a}_n z f_{n+1}. \end{aligned}$$

Using $f_n b_n = g + o(1)$, $f_{n+1} b_{n+1} = g + o(1)$, which hold uniformly on compact subsets of $\mathbb{D}$ by Lemma 4.1, we obtain, multiplying (4.7) by f_{n+1}, that

$$\begin{aligned} &zg\{\bar{a}_n z f^2_{n+1} + a_n g + z\,|a_n|^2\, g f_{n+1}\} \\ &\quad = a_n g + |a_n|^2\, f_{n+1} + \bar{a}_n z f^2_{n+1} + o(1), \qquad n \to +\infty, \end{aligned}$$

which can be rewritten follows

$$(zg - 1)\{\bar{a}_n z f^2_{n+1} + |a_n|^2\,(1+zg)\, f_{n+1} + a_n g\} = o(1), \qquad n \to +\infty.$$

Since $g \in \mathscr{B}$, the function $zg - 1$ is bounded away from zero on any compact subset of $\mathbb{D}$. It follows that

(4.8) $$\bar{a}_n z f^2_{n+1} + |a_n|^2\,(1+zg)\, f_{n+1} + a_n g = o(1), \qquad n \to +\infty,$$

uniformly on compact subsets of $\mathbb{D}$.

Let us now put $z = 0$ in (4.2). Then by (1.3) and (1.11) we obtain

(4.9) $$\lim_n \, a_n \bar{a}_{n-1} = -g(0) \neq 0.$$

It follows that there exists a positive integer N such that

(4.10) $$|a_n| > c > 0, \qquad c = |g(0)|/2,$$

for every n, $n > N$.

Multiplying (4.8) by $z\bar{a}_n\,|a_n|^{-2}$, we obtain (4.5), since the a_n's satisfy (4.10) Multiplying (4.8) by $a_n b_{n+1}^2\,|a_n|^{-2}$, we obtain (4.6), since the holomorphic function g satisfies $g(0) \neq 0$. ∎

The quadratic equation (4.4) has two roots, which call be written down explicitly:

$$\begin{aligned} X_{1,n} &= \tfrac{1}{2}\left\{-|a_n|\,(1+zg)+\sqrt{|a_n|^2\,(1+zg)^2-4zg}\right\} \\ &= \tfrac{1}{2}\,(1+zg)\left\{\sqrt{|a_n|^2-\Omega(z)}-|a_n|\right\}, \\ X_{2,n} &= -\tfrac{1}{2}\,(1+zg)\left\{\sqrt{|a_n|^2-\Omega(z)}+|a_n|\right\}. \end{aligned} \tag{4.11}$$

In (4.11) we take, the branch of the square root $\sqrt{w}$ assuming positive values for positive w. We set

$$\Omega(z) = \frac{4zg}{(1+zg)^2}. \tag{4.12}$$

LEMMA 4.3. *Let $g \in \mathscr{B}$ be the Schur function of $\nu \in \mathscr{P}$ and Ω be defined by* (4.12). *Then*

$$\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\,d\nu(\zeta) = \frac{1+zg}{1-zg} = \frac{1}{\sqrt{1-\Omega}}. \tag{4.13}$$

In particular, Ω maps $\mathbb{D}$ into $\mathbb{C}\setminus[1,+\infty)$.

Proof. It follows from the elementary identity

$$\Omega(z) = \frac{4zg}{(1+zg)^2} = \frac{(1+zg)^2-(1-zg)^2}{(1+zg)^2} = 1-\left(\frac{1-zg}{1+zg}\right)^2. \quad ∎$$

It is useful to notice that implicitly we already used (4.13) in the proof of Proposition 3.5 with $\Omega(z) = -4a^2 z(1-z)^{-2}$.

Since $\Omega(0) = 0$ and the a_n's satisfy (4.10) for $n > N$, we conclude that $X_{1,n}$ and $X_{2,n}$ (for $n > N$) are well-defined holomorphic functions in the disk $\{z\colon |z| \leqslant \rho\}$, where ρ is a small positive number. The functions $X_{1,n}$ and $X_{2,n}$ differ by their values at $z = 0$:

$$X_{1,n}(0) = 0, \qquad X_{2,n}(0) = -|a_n|. \tag{4.14}$$

Let us now put $X = X_{1,n}$ in (4.4) and subtract the identity obtained from (4.5). We obtain that the following relation

$$\left(z\frac{\bar{a}_n}{|a_n|}\,f_{n+1}-X_{1,n}\right)\left\{z\frac{\bar{a}_n}{|a_n|}\,f_{n+1}+X_{1,n}+|a_n|\,(1+zg)\right\} = o(1) \tag{4.15}$$

holds uniformly on the closed disk $z\colon |z| \leqslant \rho$, as $n \to +\infty$. Notice that

$$X_{1,n} = -\frac{2zg}{|a_n|\,(1+zg)} \frac{1}{\sqrt{1-\frac{1}{|a_n|^2}\Omega+1}};$$

see (4.11) and (4.12). Hence, it follows from (4.10) that $|X_{1,n}| < c/6$ and $|z| < c/6$ if $|z| \leqslant \rho$ for a sufficiently small positive ρ, where c satisfies (4.10). Since $f_{n+1} \in \mathscr{B}$ and $g \in \mathscr{B}$, the modulus of the braces in (4.15) exceeds $c - 3c/6 = c/2$ in $\{z\colon |z| \leqslant \rho\}$. Observing that $X_{1,n}(0) = 0$, we obtain from (4.15) Schwarz's lemma that

$$z\frac{\bar{a}_n}{|a_n|} f_{n+1}(z) = X_{1,n}(z)(1+o(1)), \qquad n \to +\infty, \tag{4.16}$$

uniformly in $\{|z|\colon |z| \leqslant \rho\}$.

Lemma 4.4. *Let σ be a probability measure on $\mathbb{T}$ satisfying* (4.1) *with such a measure ν that its Schur function $g = \mathscr{H}(\nu)$ does not vanish at $z = 0$. Let $\{a_n\}_{n \geqslant 0}$ be the Geronimus parameter of σ, $\lambda = -g(0)/|g(0)|$, and $X_{1,n}$ be defined by* (4.11). *Then*

$$X_{1,n-1}(z) = \lambda z \frac{X_{1,n} + |a_n|}{1+|a_n|\,X_{1,n}(z)}(1+o(1)), \qquad n \to +\infty, \tag{4.17}$$

uniformly in $\{z\colon |z| \leqslant \rho\}$ for some small positive ρ.

Proof. Since $|X_{1,n}(z)| < c/6$ in $\{z\colon |z| \leqslant \rho\}$, we obtain that

$$|X_{1,n}(z) + |a_n|| > c/2, \qquad |1+|a_n|\,X_{1,n}(z)| > 1 - c/6; \tag{4.18}$$

see (4.10) and (4.11). By (1.3) we have

$$f_n(z) = \frac{zf_{n+1} + |a_n|}{1+\bar{a}_n z f_{n+1}} = \frac{a_n}{|a_n|} \frac{\frac{\bar{a}_n}{|a_n|} zf_{n+1}(z) + |a_n|}{1+|a_n|\frac{\bar{a}_n}{|a_n|} zf_{n+1}(z)}.$$

Using (4.16) and (4.18), we obtain that

(4.19)

$$\frac{\bar{a}_n}{|a_n|} f_n(z) = \frac{X_{1,n}(z)(1+o(1)) + |a_n|}{1+|a_n|\,X_{1,n}(z)(1+o(1))} = \frac{X_{1,n}(z) + |a_n|}{1+|a_n|\,X_{1,n}(z)}(1+o(1)),$$

uniformly on $\{z\colon |z| \leqslant \rho\}$ as $n \to \infty$. Finally,

$$\frac{\bar a_n}{|a_n|} = \frac{\bar a_{n-1}}{|a_{n-1}|}\frac{\bar a_n a_{n-1}}{|a_n a_{n-1}|} = \frac{\bar a_{n-1}}{|a_{n-1}|}\left\{-\frac{\bar g(0)}{|g(0)|}\right\}(1+o(1)) \tag{4.20}$$
$$= \bar\lambda \frac{\bar a_{n-1}}{|a_{n-1}|}(1+o(1)),$$

by (4.9). Multiplying (4.19) by λz and replacing $\bar a_n/|a_n|$ with the right-hand side of (4.20), we obtain (4.17) from (4.16). ▮

Given $a,\ a' \in [0,\ 1)$ we define $f_{a,a'}$ to be the function in $\mathscr{B}$ with the Schur parameters

$$\mathscr{S} f_{a,a'} = (a,\ a',\ a,\ a',\ \ldots).$$

Let us obtain an explicit formula for $f_{a,a'}$. It is clear from (1.3) that $f_{a,a'}$ satisfies the equation

$$X = \frac{z(zX+a')+a(1+a'zX)}{(1+a'zX)+az(zX+a')},$$

which is equivalent to the quadratic equation

$$z(az+a')\,X^2-(z^2-1)\,X-(a'z+a)=0.$$

Since $f_{a,a'}(0)=a$, we conclude that

$$f_{a,a'}(z) = \frac{z^2-1}{2z(az+a')}\left\{1-\sqrt{1+\frac{4z(az+a')(a'z+a)}{(z^2-1)^2}}\right\}, \tag{4.21}$$

where $\sqrt{1}=1$. To simplify the notations we put

$$\Omega_{a,a'}(z) = -\frac{4z(az+a')(a'z+a)}{(z^2-1)^2}. \tag{4.22}$$

Then

$$f_{a',a}(z) = \frac{z^2-1}{2z(a'z+a)}\{1-\sqrt{1-\Omega_{a,a'}(z)}\}. \tag{4.23}$$

Observing that $\mathscr{S} f_{a,1} = (a,\ 1)$, one can cheek by a direct calculation that

$$f_{a,1} = \frac{z+a}{1+az} = \frac{z^2-1}{2z(az+1)}\{1-\sqrt{1-\Omega_{a,1}(z)}\}. \tag{4.24}$$

Indeed, the second equality of (4.24) follows from the elementary identity

$$(1-z^2)^2+4z(az+1)(z+a)=(z^2+2az+1)^2.$$

THEOREM 4.5. *Let σ be a probability measure on $\mathbb{T}$ satisfying* (4.1) *with such a measure ν that $g=\mathscr{H}(\nu)$ does not vanish at $z=0$. Let $\{a_n\}_{n\geqslant 0}$ be the Geronimus parameters of σ, $\lambda=-g(0)/|g(0)|$. Then the following limits exist*

(4.25)
$$\lim_n |a_{2n}|=a, \qquad \lim_n |a_{2n-1}|=a', \qquad \lim_n \frac{a_{2n}}{a_{2n-1}}=\frac{a}{a'}\lambda, \qquad \lim_n \frac{a_{2n+1}}{a_{2n}}=\frac{a'}{a}\lambda,$$

and

$$\Omega(z)\overset{\text{def}}{=}\frac{4zg}{(1+zg)^2}=-\frac{4\lambda z(a'z+a)(a\lambda z+a')}{(\lambda^2z^2-1)^2}. \tag{4.26}$$

Proof. Let Λ be any infinite subset of $\mathbb{Z}_+$ such that $\lim_{n\in\Lambda}|a_n|=a'$ for some $a'\in(0,1]$. Then, by (4.9), $\lim_{n\in\Lambda}|a_{n-1}|$ exists and equals a, where $aa'=|g(0)|$. It follows from (4.11) that

$$\begin{aligned} \lim_{n\in\Lambda} X_{1,n-1}(z)&=\frac{1+zg}{2}\{\sqrt{a^2-\Omega}-a\}=\frac{\sqrt{a^2-\Omega}-a}{\sqrt{1-\Omega}+1}\overset{\text{def}}{=}\lambda z\xi_a(\lambda z)\\ \lim_{n\in\Lambda} X_{1,n}(z)&=\frac{\sqrt{a'-\Omega}-a'}{\sqrt{1-\Omega}+1}\overset{\text{def}}{=}\lambda z\xi_{a'}(\lambda z), \end{aligned} \tag{4.27}$$

uniformly in $\{z\colon |z|\leqslant\rho\}$, where ρ is small and positive. By Lemma 4.4 we obtain that

$$\xi_a(z)=\frac{z\xi_{a'}(z)+a'}{1+a'z\xi_{a'}(z)}, \tag{4.28}$$

in the disk $\{z\colon |z|\leqslant\rho\}$.

First let us consider the case $a'=1$. Then (4.27) implies that $\xi_a\equiv 1$. It follows from the definition of ξ_a (see (4.26)) that

$$\sqrt{a^2-\Omega}=a+\lambda z+\lambda z\sqrt{1-\Omega},$$

which shows that $\sqrt{1-\Omega}$ satisfies the quadratic equation

$$((\lambda z)^2-1)\,X^2+2\lambda z(a+\lambda z)\,X+(1+(\lambda z)^2+2a\lambda z)=0.$$

Taking into account that $\Omega(0)=0$ and $\sqrt{1}=1$, we obtain that

$$\sqrt{1-\Omega}=\frac{(\lambda z)^2+2a\lambda z+1}{1-(\lambda z)^2}.$$

By Lemma 4.3 we obtain

$$zg=\frac{1-\sqrt{1-\Omega}}{1+\sqrt{1-\Omega}}=-\frac{\lambda z(\lambda z+a)}{1+a\lambda z}.$$

It follows that this identity holds in the disk $\mathbb{D}$ and that $\Omega=\Omega_{a,1}(\lambda z)$. The case $a=1$ reduces to the case $a'=1$ if we denote by Λ the shifted set $\Lambda-1$.

Suppose now that a, $a'\in(0,1)$. Then in addition to (4.27) we have

$$\xi_{a'}(z)=\frac{z\xi_a(z)+a}{1+az\xi_a(z)}. \tag{4.28}$$

Substituting (4.27) into (4.28), we obtain that $\xi_{a'}$ satisfies the quadratic equation (4.21). Since $\xi_{a'}(0)=a$, we conclude that $\xi_{a'}(z)=f_{a,a'}(z)$ in the disk $\{z\colon |z|\leqslant\rho\}$ and therefore in $\mathbb{D}$ by the uniqueness theorem for analytic functions. Hence $\xi_a(z)=f_{a,a'}(z)$. By (4.16), we obtain that

$$\sqrt{a^2-\Omega}=a+w\xi_a(w)+w\xi_a(w)\sqrt{1-\Omega},\qquad w=\lambda z,$$

which shows that $\sqrt{1-\Omega}$ satisfies the quadratic equation

$$(w^2\xi_a^2-1)\,X^2+2w\xi_a(a+w\xi_a)\,X+(1+2aw\xi_a+w^2\xi_a^2)=0.$$

It follows that

$$\sqrt{1-\Omega}=\frac{w^2\xi_a^2+2aw\xi_a+1}{1-w^2\xi_a^2}$$

and, therefore, by Lemma 4.3

$$zg=-w\xi_a(w)\,\frac{w\xi_a(w)+a}{1+aw\xi_a(w)}=-wf_{a,a'}(w)\,\frac{wf_{a,a'}+a}{1+awf_{a',a}(w)},\qquad w=\lambda z.$$

Hence

$$zg(z)=-wf_{a',a}(w)\,f_{a,a'}(w),\qquad w=\lambda z. \tag{4.29}$$

Multiplying (4.21) and (4.22), we obtain

$$-wf_{a',a}(w)\,f_{a,a'}(w)=\frac{(1-\sqrt{1-\Omega_{a,a'}(w)})^2}{\Omega_{a,a'}(w)}=\frac{1-\sqrt{1-\Omega_{a,a'}(w)}}{1+\sqrt{1-\Omega_{a,a'}(w)}}.$$

Therefore $\omega(z)=\Omega_{a,a'}(\lambda z)$, which is in fact (4.25).

Suppose now that $a\neq a'$. Then Ω has two zeros $-\bar{\lambda}a/a'$ and $-\bar{\lambda}a'/a$, one of which must be in $\mathbb{D}$ and coincide with the zero of g; see (3.26). Since $aa'=|g(0)|$, we may conclude that the pair (a,a') is uniquely determined by g. This implies that as soon as the limit $\lim_{n\in\Lambda}|a_n|$ exists for some infinite subset Λ of $\mathbb{Z}_+$, it must be equal either to a or to a'. Let $\varepsilon=|a-a'|/3$. Then

$$\operatorname{Card}\{n:||a_n|-a|\geqslant\epsilon,\ ||a_n|-a'|\geqslant\epsilon\}<+\infty.$$

It follows that there exists a positive integer N such that for every $n\geqslant N$ either $||a_n|-a|<\epsilon$ or $||a_n|-a'|<\epsilon$.

Let us consider now the set $\Lambda=\{n\in\mathbb{Z}_+:||a_n|-a|<\varepsilon\}$, which cannot be finite by the definition of a. Since the limit of $|a_n|$ along any subset of Λ cannot be equal to a', due to our choice of ε, we obtain that the limit $\lim_{n\in\Lambda}|a_n|$ exists and equals a, which implies that $\lim_{n\in\Lambda}|a_{n-1}|=a'$.

It follows that there exists a positive number $N_1\geqslant N$ such that the sets $\Lambda\cap[N_1,+\infty)$ and $(\Lambda-1)\cap[N_1,+\infty)$ complete each other in $[N_1,+\infty)$. Since these sets differ by a shift, we conclude that one of them is the subset of the even numbers. This obviously implies the first two equalities in (4.25) (up to the order of a and a'). To obtains the third equality of (4.25), we may write

(4.30)
$$\lim_n\frac{a_{2n}}{a_{2n-1}}=\lim_n\frac{a_{2n}\bar{a}_{2n-1}}{|a_{2n-1}|^2}$$
$$=-\frac{g(0)}{a'^2}=-\frac{g(0)}{|g(0)|}\,\frac{a}{a'}=\lambda\frac{a}{a'},$$

using (4.10), the identity $aa'=|g(0)|$, and the definition of λ.

Suppose now that $a=a'$. Then from (4.26), which we have already proved, we have

$$\Omega(z)=-\frac{4\lambda za^2}{(1-\lambda z)^2},$$

and $g(z)=-\lambda f_a^2(\lambda z)$ by (4.29). It follows that the pair (a,λ) can be uniquely restored by the identity $g(0)=-\lambda a^2$. Hence, if the limit $\lim_{n\in\Lambda}|a_n|$ exists for some infinite set Λ, then it must be equal to a. This obviously implies that $\lim_n |a_n| = a$. It follows from (4.30) that $\{a_n\}_{n\geqslant 0}$ satisfies the López condition. ▌

Proof of Theorem B. The implication (L) $\Rightarrow$ (1.30) follows by Theorem 3.1 and by Proposition 3.5.

Suppose that (1.30) holds for some σ and some pair (a, λ). Applying Theorem 3.1 to any measure satisfying the López condition with (a, λ), we obtain that $g = -\lambda f_a^2(\lambda z)$ is the Schur function of $\nu_{a,\lambda}$. Since $g(0) = -\lambda a^2 \neq 0$, Theorem 4.5 says that σ satisfies (L). ▌

5. EQUILIBRIUM MEASURES FOR $J(a, a') \cup \bar{J}(a, a')$

The following theorem extends Theorem B to measures $\mu_{a,a',\lambda}$.

THEOREM 5.1. *Let* $\sigma \in \mathscr{P}$, $\{\varphi_n\}_{n\leqslant 0}$ *be the orthogonal polynomials in* $L^2(d\sigma)$, $\{a_n\}_{n\geqslant 0}$ *the Geronimus parameters of* σ. *Let* $\mu_{a,a'}$ *be defined by* (1.29) *with* $a, a' \in (0, 1]$ *and* $\mu_{a,a',\lambda}$ *be the rotation of* $\mu_{a,a'}$ *by* λ, $\lambda \in \mathbb{T}$. *Then*

$$(5.1) \qquad (*)-\lim_n |\varphi_n|^2\, d\sigma = d\mu_{a,a',\lambda}$$

if and only if the sequence $\{a_n\}_{n\geqslant 0}$ *satisfies* (4.25) *either with* (a, a', λ) *or with* (a', a, λ). *The Schur function of* $\mu_{a,a',\lambda}$ *is given by*

$$(5.2) \qquad \mathscr{H}(\mu_{a,a',\lambda}) = -\lambda f_{a,a'}(\lambda z)\, f_{a',a}(\lambda z),$$

where $f_{a,a'}(z)$ *is defined by* (4.21).

Proof. It follows from (4.25) that

$$(5.3) \qquad \begin{aligned} &\lim_n \frac{\bar{a}_{2n}}{|a_{2n}|} a_{2n+k} = \lim_n |a_{2n}| \lim_n \frac{a_{2n+k}}{a_{2n}} = \begin{cases} a\lambda^k & \text{if } k \text{ is even,} \\ a'\lambda^k & \text{if } k \text{ is odd,} \end{cases} \\ &\lim_n -\frac{a_{2n}}{|a_{2n}|} \bar{a}_{2n-k} = \lim_n |a_{2n}| \lim_n -\overline{\left(\frac{a_{2n-k}}{a_{2n}}\right)} = \begin{cases} -a\lambda^k & \text{if } k \text{ is even,} \\ -a'\lambda^k & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

By Theorem 1.1 and (3.2) this implies that

$$(5.4) \qquad \lim_n \frac{\bar{a}_{2n}}{|a_{2n}|} f_{2n+1}(z) = \lambda f_{a',a}(\lambda z); \qquad \lim_n \frac{a_{2n}}{|a_{2n}|} b_{2n+1}(z) = -f_{a,a'}(\lambda z).$$

Replacing $2n$ by $2n+1$ in (5.3), we obtain similar formulae with a and a' transposed, which together with Theorem 1 and (3.2) imply

(5.5)

$$\lim_n \frac{\bar{a}_{2n+1}}{|a_{2n+1}|} f_{2n+2}(z) = \lambda f_{a,a'}(\lambda z); \qquad \lim_n \frac{a_{2n+1}}{|a_{2n+1}|} b_{2n+2}(z) = -f_{a',a}(\lambda z).$$

It is clear from (5.4) and (5.5) that

$$\lim_n f_n(z)\, b_n(z) = -\lambda f_{a,a'}(\lambda z)\, f_{a',a}(\lambda z) \tag{5.6}$$

uniformly on compact subsets of $\mathbb{D}$. Hence by Theorem 1.4 the limit in the left-hand side of (5.1) exists and equals $d\mu$, $\mu \in \mathscr{P}$.

To show that μ is in fact $\mu_{a,a',\lambda}$, we apply Lemma 4.3, which says that

$$\mathscr{H}(\mu) = \frac{1}{\sqrt{1-\Omega_{a,a'}(\lambda z)}}, \tag{5.7}$$

where $\Omega_{a,a'}$ is defined by (4.22). A direct calculation shows

$$1-\Omega_{a,a'}(\exp(i\theta)) = \frac{(\cos\theta - \cos\theta_2)(\cos\theta_1 - \cos\theta)}{\sin^2\theta}. \tag{5.8}$$

Here $\cos\theta_1$ and $\cos\theta_2$ are defined as in Section 3.

Suppose first that $\cos\theta_1 \neq \cos\theta_2$. Then the rational function $1-\Omega_{a,a'}$ has only simple zeros on $\mathbb{T}$. It follows from (5.7) that μ is absolutely continuous. Without loss of generality we may assume that $\lambda = 1$. Combining (5.7) with (5.8), we obtain that $\mu = \mu_{a,a'}$, where $\mu_{a,a'}$ is defined by (1.29). Thus (5.1) holds if $\max(a, a') < 1$.

Suppose now that $\max(a, a') = 1$. Then $t = \exp(i\theta_1)$ is the zero of the second order for $1-\Omega_{a,a'}$ on $\mathbb{T}$ and $1-\Omega_{a,a'}$ is negative on $\mathbb{T} \setminus (\{t\} \cup \{\bar{t}\})$. Again we may assume $\Lambda = 1$. Hence μ is a convex combination of the point masses $\delta_{\{t\}}$ and $\delta_{\{\bar{t}\}}$, $t = \exp(i\theta_1)$. By symmetry, the coefficients in this convex combination must be equal. It follows that $\mu = \mu_{a,a'}$ in this case too.

Since $\mathscr{H}(\mu_{a,a',\lambda})(0) = -\lambda a a' \neq 0$, the implication (5.1) $\Rightarrow$ (4.25) follows from Theorem 4.5. ∎

Observe that for $a = a'$ (4.25) reduces to the López condition (L). Hence the following corollary is immediate by Theorems 3.1 and 5.1.

COROLLARY 5.2. *For any* $a \in [0, 1]$ *we have* $\nu_{a,\lambda} = \mu_{a,a,\lambda}$.

It follows that to obtain an explicit formula for the density of the equilibrium distribution ν_a for the arc Δ_α, $\alpha = 2 \arcsin a$, we can just put $a = a'$

in (1.29). Since $\cos\theta_2=-1$, $\cos\theta_1=1-2a^2$, we obtain that $\sin(\theta_1/2)=a$. Therefore $J(a,a)\cup\bar{J}(a,a)=\Delta_\alpha$ and

$$p_a(\theta)\overset{\text{def}}{=}\frac{|\sin\theta|}{\sqrt{(\cos\theta+1)(\cos\alpha-\cos\theta)}}=\frac{\left|\sin\frac{\theta}{2}\right|}{\sqrt{\cos^2\frac{\alpha}{2}-\cos^2\frac{\theta}{2}}}\tag{5.9}$$

for $\alpha\leqslant|\theta|\leqslant\pi$, $d\nu_a=p_a\,dm$.

THEOREM 5.3. *Let* $\max(a,a')<1$. *Then the probability measure* $\mu_{a,a'}$ *is the equilibrium distribution with respect to the logarithmic kernel for* $J(a,a')\cup\bar{J}(a,a')$.

Proof. For any $\mu\in\mathcal{P}$ with $g=\mathcal{H}(\mu)$ the formulae

$$\int_{\mathbb{T}}\frac{d\mu(\zeta)}{\zeta-z}=\frac{g(z)}{1-zg(z)},\qquad \frac{d}{dz}\int_{\mathbb{T}}\log\frac{1}{\zeta-z}\,d\mu(\zeta)=\int_{\mathbb{T}}\frac{d\mu(\zeta)}{\zeta-z}$$

imply that

$$\int_{\mathbb{T}}\log\frac{1}{|\zeta-z|}\,d\mu(\zeta)=\Re\int_0^z\frac{g(\zeta)}{1-\zeta g(\zeta)}\,d\zeta.\tag{5.10}$$

By Lemma 4.3 we have

$$\frac{g}{1-zg}=\frac{1}{2z}\left\{\frac{1}{\sqrt{1-\Omega}}-1\right\}.\tag{5.11}$$

Let us apply (5.10) and (5.11) to $\mu_{a,a'}$ with $\max(a,a')<1$. Since $\Omega(0)=0$ and $\sqrt{x^2}=x$ for $x>0$, we have

$$\begin{aligned}\sqrt{1-\Omega_{a,a'}(z)}&=\frac{1}{1-z^2}\sqrt{z^4+4aa'z^3+(4a'^2+4a^2-2)\,z^2+4aa'z+1}\\&=\frac{z}{1-z^2}\sqrt{\left(z+\frac{1}{z}\right)^2+4aa'\left(z+\frac{1}{z}\right)+4(a^2+a'^2-1)}.\end{aligned}$$

Let us observe that if we put

$$w(z)=\frac{1}{2}\left(z+\frac{1}{z}\right),$$

then $\sqrt{w^2} = w$ on a neighborhood of $w = \infty$. It follows that

$$\frac{dz}{2z\sqrt{1-\Omega_{a,a'}(z)}} = -\frac{dw}{2\sqrt{(w+aa')^2-(1-a^2)(1-a'^2)}},$$

which implies that

$$\text{(5.12)} \quad \int \frac{1}{2z}\left\{\frac{1}{\sqrt{1-\Omega_{a,a'}}}-1\right\} dz = -\frac{1}{2}\log[zw+aa'+z\sqrt{(w+aa')^2-(1-a^2)(1-a'^2)}].$$

Observe that the function in the brackets at the right-hand side of (5.12) assumes the value 1 at $z = 0$. Therefore, we obtain by the Newton–Leibnitz formula that

$$\text{(5.13)} \quad \int_{\mathbb{T}} \log\frac{1}{|\zeta - z|}\, d\mu_{a,a'} = -\frac{1}{2}\log|zw(z)+aa'z+z\sqrt{(w+aa')^2-(1-a^2)(1-a'^2)}\,|.$$

It is easy to see that the expression under the logarithm in the right-hand side of (5.13) is equal to

$$\text{(5.14)} \quad |\cos\theta + aa' + \sqrt{(\cos\theta+aa')^2-(1-a^2)(1-a'^2)}\,|$$

if $z = \exp(i\theta)$, $\theta \in [-\pi, \pi]$. It follows from the definition of $J(a, a')$ that

$$-\sqrt{(1-a^2)(1-a'^2)} \leqslant \cos\theta + aa' \leqslant \sqrt{(1-a^2)(1-a'^2)}$$

if $\exp(i\theta) \in J(a, a') \cup \bar{J}(a, a')$. This observation implies that the function (5.14) takes the constant value $\sqrt{(1-a^2)(1-a'^2)}$ on $J(a, a') \cup \bar{J}(a, a')$. Hence the logarithmic potential (5.13) equals $-\log\sqrt[4]{(1-a^2)(1-a'^2)}$ on $J(a, a') \cup \bar{J}(a, a')$, which implies that $\mu_{a,a'}$ is the equilibrium distribution for $J(a, a') \cup \bar{J}(a, a')$, since $\mu_{a,a'} \in \mathscr{P}$. This also implies that

$$\text{(5.15)} \quad C_\ell(J \cup \bar{J}) = \sqrt[4]{(1-a^2)(1-a'^2)},$$

where C_ℓ is the logarithmic capacity; see [31, Chap. 5]. ∎

Proof of Theorem C. It follows from Theorems 5.1 and 5.3. ∎

Proof of Theorem D. Let us observe that the first formula (1.35) can be written as follows

$$2aa' = -a_{2n}(1-a_{2n-1}) + a_{2n-2}(1+a_{2n-1}),$$

which implies two identities

$$
(5.16)\qquad \begin{aligned} 2(1+aa') &= (1-a_{2n})(1-a_{2n-1})+(1+a_{2n-2})(1+a_{2n-1}) \\ 2(1-aa') &= (1+a_{2n})(1-a_{2n-1})+(1-a_{2n-2})(1+a_{2n-1}). \end{aligned}
$$

By the second formula in (1.35), we have

$$
(5.17)\qquad (1-a^2)(1-a'^2) = (1-a_{2n})(1-a_{2n-1})(1+a_{2n+1})(1+a_{2n}).
$$

Let us consider an auxiliary sequence

$$
x_n = (1-a_{2n})(1-a_{2n-1}), \qquad n = 1, 2, \ldots, \; x_0 = 1+aa'.
$$

Combining the first identity (5.16) with (5.17), we obtain

$$
x_{n+1} = 2(1+aa') - \frac{(1-a^2)(1-a'^2)}{x_n}, \qquad n = 0, 1, \ldots.
$$

The equation $X^2 - 2(1+aa')\,X + (1-a^2)(1-a'^2) = 0$ has two different roots

$$
r = (1-a)(1-a'), \qquad r' = (1+a)(1+a')
$$

provided $1 > \max(a, a') > 0$. Next, $x_0 = 1+aa' \in (r, r')$. By Lemma 3.7, we obtain that $\{x_n\}_{n\geqslant 0}$ is an increasing sequence such that

$$
(5.18)\qquad x_n = (1+a)(1+a') - c_1\left(\frac{(1-a)(1-a')}{(1+a)(1+a')}\right)^n (1+o(1)), \qquad n \to +\infty,
$$

for some positive constant c_1.

Let us consider another auxiliary sequence

$$
y_n = (1+a_{2n})(1-a_{2n-1}), \qquad n = 1, 2, \ldots, \quad y_0 = 1-aa'.
$$

Combining the second identity in (5.16) with (5.17), we obtain

$$
y_{n+1} = 2(1-aa') - \frac{(1-a^2)(1-a'^2)}{y_n}, \qquad n = 0, 1, \ldots.
$$

The equation $X^2 - 2(1-aa')\,X + (1-a^2)(1-a'^2) = 0$ has two different roots

$$
r = (1+a)(1-a'), \qquad r' = (1-a)(1+a')
$$

provided $1 > \max(a, a') > 0$. Assuming (without loss of generality that $a' > a$, we have $r < r'$. By Lemma 3.7 obtain that $\{y_n\}_{n \geqslant 0}$ is an increasing sequence such that

(5.19)
$$y_n = (1-a)(1+a') - c_2 \left(\frac{(1+a)(1-a')}{(1-a)(1+a')} \right)^n (1+o(1)), \qquad n \to +\infty,$$

for some positive c_2.

Suppose that $a > 0$. Then by (5.18) and (5.19) we have,

(5.20)
$$\begin{aligned} a_{2n} &= \frac{y_n - x_n}{y_n + x_n} = -a - \frac{c_2}{2} \frac{1+a}{1+a'} \left(\frac{(1+a)(1-a')}{(1-a)(1+a')} \right)^n (1+o(1)) \\ a_{2n+1} &= 1 - \frac{y_{n+1} + x_{n+1}}{2} = -a' + \frac{c_2}{2} \left(\frac{(1+a)(1-a')}{(1-a)(1+a')} \right)^n (1+o(1)). \end{aligned}$$

For $a = 0$ we obtain obviously that $a_{2n} = 0$, $n = 0, 1, \ldots$.

$$a_{2n+1} = 1 - x_{n+1} = -a' + c_1 \left(\frac{1-a'}{1+a'} \right)^{n+1} (1+o(1)).$$

We observe that $c_1 = c_2$ if $a = 0$.

Suppose that $\max(a, a') = 1$. Then $a_{2n+1} = -1$, $n = 0, 1, \ldots$, by (1.35) and therefore $a_{2n} = -a$, $n = 0, 1, \ldots$, by the first identity (1.35). This completes the proof of D. ▮

If we compare (5.20) with (1.33), we find that we cannot take $a = a'$ in (3.15). In order to study this phenomenon as well as to find the value c_2 in (5.20) we consider another proof of Theorem D based on the theory of Tchebysheff polynomials.

Proof of Theorem D (Tchebysheff's approach). Without loss of generality, assume that $0 \leqslant a < a' < 1$. Then the measure $\mu_{a,a'}$ is absolutely continuous and its image under the projection $z \to \frac{1}{2}(z + 1/z)$ is $\pi^{-1} w_{a,a'}\, dx$ see (1.25) and (1.26). It is clear from the second equality (1.26) that the monic orthogonal polynomials P_n in $L^2(w_{a,a'})$ can be written down explicitly in terms of the monic Tchebysheff polynomials of the first kind [23]

(5.21)
$$\dot{T}_n(x) = \frac{1}{2^n} \{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2-1})^n\}$$

as follows

(5.22)
$$P_n(x) = \omega_1^{n/2} \dot{T}_n \left(\frac{x + aa'}{\sqrt{\omega_1}} \right).$$

We are going to apply (1.38) and (1.39). Notice that

$$y_{-1} \stackrel{\text{def}}{=} \frac{-1+aa'}{\sqrt{\omega_1}} < -1 < 1 < \frac{1+aa'}{\sqrt{\omega_1}} \stackrel{\text{def}}{=} y_1.$$

Assuming that $y \notin [-1, 1]$, we obtain from (5.21) that

(5.23)

$$\frac{\dot{T}_{n+1}(y)}{\dot{T}_n(y)} = \frac{1}{2}(y+\sqrt{y^2-1})\{1-(1-q(y))\,q(y)^n\,(1+o(1))\}, \qquad n \to +\infty,$$

where

$$q(y) = \frac{1}{(y+\sqrt{y^2-1})^2} \in \mathbb{D}.$$

We consider in the above formula the branch of algebraic function $\sqrt{y^2-1}$ defined in $\hat{\mathbb{C}} \setminus [-1, 1]$ by $\sqrt{y^2-1} \sim y$, $y \to \infty$. Observing that

$$y_1 + \sqrt{{y_1}^2-1} = \frac{(1+a)(1+a')}{\sqrt{(1-a^2)(1-a'^2)}},$$

$$q(y_1) = \frac{(1-a)(1-a')}{(1+a)(1+a')},$$

$$y_{-1} + \sqrt{{y_{-1}}^2-1} = -\frac{(1-a)(1+a')}{\sqrt{(1-a^2)(1-a'^2)}},$$

$$q(y_{-1}) = \frac{(1+a)(1-a')}{(1-a)(1+a')},$$

we obtain by (1.39), (5.22), and (5.23) that

$$\begin{aligned} u_n &= \frac{P_{n+1}(1)}{P_n(1)} \\ &= \frac{1}{2}(1+a)(1+a')\{1-(1-q(y_1))\,q(y_1)^n\,(1+o(1))\}, \qquad n \to \infty, \end{aligned}$$

$$\begin{aligned} v_n &= \frac{P_{n+1}(-1)}{P_n(-1)} \\ &= \frac{1}{2}(1-a)(1+a')\{1-(1-q(y_{-1}))\,q(y_{-1})^n\,(1+o(1))\}, \qquad n \to \infty. \end{aligned}$$

We have $q(y_1) \geqslant q(y_1)$ since $a' > a$. The equality takes place only for $a = 0$.

We suppose first that $a>0$. Then

$$v_n-u_n=-a(1+a')-(a'-a)\left(\frac{(1+a)(1-a')}{(1-a)(1+a')}\right)^n(1+o(1)),$$

$$v_n+u_n=(1+a')-(a'-a)\left(\frac{(1+a)(1-a')}{(1-a)(1+a')}\right)^n(1+o(1)).$$

It follows that

$$\begin{aligned} a_{2n}&=\frac{v_n-u_n}{v_n+u_n}\\ &=-a-(a'-a)\frac{1+a}{1-a'}\left(\frac{(1+a)(1-a')}{(1-a)(1+a')}\right)^n(1+o(1)), \qquad n\to\infty,\\ a_{2n+1}&=1-u_{n+1}-v_{n+1}\\ &=-a'+(a'-a)\left(\frac{(1+a)(1-a')}{(1-a)(1+a')}\right)^{n+1}(1+o(1)), \qquad n\to\infty. \end{aligned} \tag{5.24}$$

For $a=0$ we have $a_{2n}=0$, $n=0, 1, \ldots$, and

$$a_{2n+1}=-a'+2a'\left(\frac{1-a'}{1+a'}\right)^{n+1}(1+o(1)).$$

This completes the proof. Comparing (5.24) with (5.20), we find that $c_2=2(a-a')$. ∎

6. CLASSIFICATION OF WEAKLY ASYMPTOTIC MEASURES

To prove Theorem E we need two lemmas.

LEMMA 6.1. *Let σ be a weakly asymptotic measure on $\mathbb{T}$ satisfying* (1.35)*, where $g=\mathscr{H}(\nu)$ is the Schur function of the limit measure ν such that $g(z)=z^k g_k$, $g_k(0)\neq 0$ for some integer $k\geqslant 1$. Let $\{a_n\}_{n\geqslant 0}$ be the Geronimus parameter of σ Then*

$$\lim_n a_n(\bar{a}_n a_{n+k+1}+g_k(0))=0, \qquad \lim_n a_n a_{n+s}=0, \qquad s=1, 2, \ldots, k. \tag{6.1}$$

Proof. By the definition of Schur's algorithm (see (1.3)) we have $g_j(z) = z^{k-j} g_k$, $j = 0, 1, \ldots, k$, which implies that $g_j(0) = 0$, $j = 0, \ldots, k-1$ We prove by induction that

$$\bar{a}_n f_{n+j+1}(z)\{zf_{n+1} + a_n(1+zg)\} = -a_n g_j + o(1), \qquad n \to +\infty \tag{6.2}$$

uniformly on compact subsets of $\mathbb{D}$, $j = 0, 1, \ldots, k$.

It is clear that for $j = 0$ (6.2) is nothing but, (4.8), which holds even in the case when $g(0) = 0$, as it is seen from the proof of (4.8). If we put $z = 0$ in (6.2) with $j = 0$, we obviously obtain

$$\lim_n a_n a_{n+1} = 0.$$

Suppose now that (6.2) holds for some $j < k$. Putting $z = 0$ in (6.2), we obtain that

$$\lim_n a_n a_{n+j+1} = 0. \tag{6.3}$$

Now by (1.3) we may write

$$f_{n+j+1}(z)(1 + \bar{a}_{n+j+1} z f_{n+j+2}(z)) = z f_{n+j+2}(z) + a_{n+j+1}.$$

Multiplying this identity by $\bar{a}_n\{zf_{n+1} + a_n(1+zg)\}$ and taking (6.3) into account, we obtain by the induction assumption (i.e., by (6.2)) that

$$z\bar{a}_n f_{n+j+2}(z)\{zf_{n+1} + a_n(1+zg)\} = -a_n g_j + o(1), \qquad n \to +\infty.$$

Since we assumed that $j < k$, $g_j(0) = 0$, we obtain by Schwarz's lemma that

$$\bar{a}_n f_{n+j+2}(z)\{zf_{n+1} + a_n(1+zg)\} = -a_n g_{j+1} + o(1), \qquad n \to +\infty,$$

which obviously completes the proof of (6.2).

Now (6.1) follows from (6.2) by putting $z = 0$ in (6.2) and choosing $j = 0, 1, \ldots, k$. ∎

LEMMA 6.2. *Let $\{a_n\}_{n \geqslant 0}$ be a sequence in $\mathbb{D}$ such that there exist a positive integer $d \geqslant 2$ and a complex number $c \neq 0$ satisfying*

$$\lim_n a_n\{\bar{a}_n a_{n+d} + c\} = 0; \qquad \lim_n a_n a_{n+s} = 0, \qquad s = 1, \ldots, d-1.$$

Then

$$\lim_{n \not\equiv r \,(\mathrm{mod}\, d)} a_n = 0, \qquad \varliminf_{n \equiv r \,(\mathrm{mod}\, d)} |a_n| > 0, \tag{6.4}$$

for some integer r, $0 \leqslant r < d$.

Proof. Let $\Lambda(\varepsilon)=\{n\in\mathbb{Z}_+ : |a_n|<\varepsilon\}$, $\varepsilon>0$. By our assumption for every ε, $0<\varepsilon<|c|/2$, there is an integer $N(\varepsilon)$ such that

$$|a_n|\,|\bar{a}_n a_{n+d}+c|<\varepsilon\frac{|c|}{2} \tag{6.5}$$

for $n>N(\varepsilon)$. If $n>N(\varepsilon)$ and $n\notin\Lambda(\varepsilon)$, then (6.5) implies

$$|\bar{a}_n a_{n+d}+c|<\frac{|c|}{2}$$

and therefore,

$$|a_{n+d}|\geqslant|\bar{a}_n a_{n+d}+c-c|\geqslant|c|-|\bar{a}|\,|a_{n+d}+c|>\frac{|c|}{2}>\varepsilon.$$

It follows that the implication

$$n\notin\Lambda(\varepsilon)\Rightarrow n+d\notin\Lambda(\varepsilon) \tag{6.6}$$

holds for every ε, $0<\varepsilon<|c|/2$, and $n\geqslant N(\varepsilon)$. Hence, the set $\{n\in\mathbb{Z}_+ : n\notin\Lambda(\varepsilon), N(\varepsilon)\}$ is an arithmetic progression $\{r+kd : k\geqslant(N(\varepsilon)-r)\,d^{-1}\}$. Since $\varepsilon\to\Lambda(\varepsilon)$ is an increasing family of subsets of $\mathbb{Z}_+$, the number r must be the same for all small ε, which obviously implies (6.4). ∎

Proof of Theorem E. Suppose that σ is a weakly asymptotic measure satisfying (1.34) with $\mathscr{H}(v)=g(z)=z^k g_k(z)$, $g_k(0)=0$, for some integer k, $k\geqslant 1$. Then by Lemmas 6.1 and 6.2 the Geronimus parameters $\{a_n\}_{n\geqslant 0}$ of σ must satisfy (6.4) with $d=k+1$ and r, $0\leqslant r\leqslant k$. It is clear that (1.34) is not violated if we drop the first r parameters of $\{a_n\}_{n\geqslant 0}$. Therefore, we may assume without loss of generality that $r=0$. Next, let σ^0 be the probability measure defined by the Geronimus parameters $\{a_n^0\}_{n\geqslant 0}$, where

$$a_n^0=\begin{cases} a_n & \text{if}\quad n\equiv 0 \pmod{k+1},\\ 0 & \text{if}\quad n\not\equiv 0 \pmod{k+1}.\end{cases}$$

We denote by f_n^0, b_n^0 the corresponding Schur functions of σ^0. Now the first formula of (3.63) and the definition of $\{a_n^0\}_{n\geqslant 0}$ imply by Theorem 1.1 that

$$f_n^0=f_n+o(1),\qquad b_n^0=b_n+o(1),\qquad n\to+\infty, \tag{6.7}$$

uniformly on subsets of $\mathbb{D}$. Hence σ^0 satisfies (1.34) with $\mathscr{H}(v)=g$.

The following simple lemma describes Schur functions with Schur parameters supported by an arithmetic progression.

LEMMA 6.3. *A function h in $\mathscr{B}$ can be presented in the form $h = H(z^d)$, $H \in \mathscr{B}$ where $d > 1$ is an integer, if and only if the Schur parameters $\{c_n\}_{n\geqslant 0}$ of h vanish ($c_n = 0$) for $n \not\equiv 0 \pmod d$.*

Proof. For $h = H(z^d)$ the result follows from the formulas (1.3) for Schur's algorithm initiated by H and followed by the substitution $z := z^d$.

Suppose now that $c_n = 0$ for $n \not\equiv \pmod d$. Then it follows front (1.3) that

$$\begin{gathered} h_0(z) \stackrel{\text{def}}{=} h(z) = \frac{z^d h_d(z) + c_0}{1 + \bar{c}_0 z^d h_d(z)}, \qquad h_d(z) = \frac{z^d h_{2d}(z) + c_d}{1 + \bar{c}_d z^d h_{2d}(z)}, \dots, \\ h_{sd} = \frac{z^d h_{sd+d}(z) + c_{sd}}{1 + \bar{c}_{sd} z^d h_{sd+d}}, \dots. \end{gathered} \tag{6.8}$$

Let us interrupt the algorithm (6.8) by replacing $h_{sd+d}(z)$ with some constant say 0 or 1. Then reading the formulae (6.8) from right to left, we see that the rational function h^0 obtained is in fact a function of z^d. On the other hand, it is known (zee, e.g. [29, (4.19), (4.21)]) that h and h^0 have matching Taylor polynomials of order $sd - d$ at $z = 0$, which implies the convergence of h^0 to h as $s \to \infty$. Hence $h = H(z^d)$ for some H in $\mathscr{B}$. ▮

Now we can complete the proof of Theorem E. Let σ^1 be the probability measure defined by the Geronimus parameters $\{a_s^1\}_{s\geqslant 0}$, $a_s^1 = a_{s(k+1)}$. It follows from the second formula (6.4) (recall that we assume $r = 0$) that $\underline{\lim}_s |a_s^1| > 0$. Let f_s^1, b_s^1 be the sth Schur functions of σ^1 and $n = s(k+1)$, $s \in \mathbb{Z}_+$. Then by Lemma 6.3 we have

$$f_n^0(z) = f_s^1(z^{k+1}), \qquad b_n^0 = z^k b_s^1(z^{k+1}).$$

By (6.4) we have

$$\lim_n z^k f_s^1(z^{k+1})\, b_s^1(z^{k+1}) = z^k g(z).$$

It follows that $g_k(z) = g^1(z^{k+1})$. Hence σ^1 satisfies (1.34) with $\mathscr{H}(v) = g^1$, $g^1(0) \neq 0$. By Theorem 4.5 the sequence $\{a_s^1\}_{s\geqslant 0}$ satisfies the López condition (L_2). By Theorem 5.1 g^1 is the Schur function of $\mu_{a,a',\lambda}$ for some choice of (a, a', λ). It follows that $v = \mu^{(k)}_{a,a',\lambda}$.

Suppose now that (1.37) holds. As above we may assume $r = 0$ and that $a_n = a_n^0$, $n \in \mathbb{Z}_+$. Then the result follows by Theorem C and by Lemma 6.3. ▮

7. THE LÓPEZ CONDITION AND MARKOFF'S THEOREM

7.1. *Perron's theorem.* We first consider connections of the López condition (L) with continued fractions which are important for the theory of orthogonal polynomials. Lemma 1.6 clarifies the relationship of (L) with the condition that either of the continued fractions (1.6) or (1.7) are limit-periodic. Obviously, we cannot claim the same with respect to the Wall continued fraction. However, this is the case if we apply to (1.8) the equivalence transformation. Recall that the continued fractions

$$q_0 + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{p_n}{q_n}\right), \qquad q_0^* + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{p_n^*}{q_n^*}\right) \tag{7.1}$$

are called equivalent if their convergents coincide [25, 2.3.1]. By [25, Theorem 2.6] the continued fractions (7.1) are equivalent if and only if there exists a sequence $\{r_n\}_{n\geqslant 0}$ of nonzero constants with $r_0 = 1$ such that

$$p_n^* = r_n r_{n-1} p_n, \quad n = 1, 2, \dots, \qquad q_n^* = r_n q_n, \quad n = 0, 1, \dots. \tag{7.2}$$

In fact, the proof can be deduced from the following elementary identity

$$\frac{r_n r_{n-1} p_n}{r_n q_n} \; \genfrac{}{}{0pt}{}{}{+} \; \frac{r_{n+1} r_n p_{n+1}}{r_{n+1} q_{n+1} + w} = \frac{r_{n-1} p_n}{q_n} \; \genfrac{}{}{0pt}{}{}{+} \; \frac{p_{n+1}}{q_{n+1} + w/r_{n+1}}.$$

We apply this transformation to (1.8) but first let us write down the convergents of (1.8). Recall [29, (1.5), (1.6)] that the *Wall polynomials* A_n, B_n associated with the parameters $\{a_n\}_{n\geqslant 0}$ determine the convergent A_n/B_n for (1.5) of order $2n$. The convergent of order $2n+1$ is given by zB_n^*/zA_n^*(see, for example, [29, Lemma 4.1]). Following the notations introduced by Khovanskii [21, 28], we can rewrite (1.8) as follows

$$\begin{array}{llllll}
a_0 + \dfrac{(1-|a_0|^2)\,z}{\bar{a}_0 z} & \genfrac{}{}{0pt}{}{}{+}\dfrac{1}{a_1} \genfrac{}{}{0pt}{}{}{+} & \dfrac{(1-|a_1|^2)\,z}{\bar{a}_1 z} & \genfrac{}{}{0pt}{}{}{+\cdots+}\dfrac{1}{a_n}\genfrac{}{}{0pt}{}{}{+} & \dfrac{(1-|a_n|^2)\,z}{\bar{a}_n z} & \genfrac{}{}{0pt}{}{}{+\cdots} \\[2ex]
-1 \quad\; 0 \quad 1 & & 2\;3 & & 2n \quad 2n+1 & \\[2ex]
\dfrac{1}{0};\; \dfrac{A_0}{B_0};\dfrac{zB_0^*}{zA_0^*}; & & \dfrac{A_1}{B_1};\dfrac{zB_1^*}{zA_1^*};\;\dots & & \dfrac{A_n}{B_n};\dfrac{zB_n^*}{zA_n^*};\;\dots. &
\end{array} \tag{7.3}$$

Here the second row is filled with the numbers of the corresponding convergents, whereas the third row lists all the convergents of the continued fraction.

The advantage of this notation is that one can control the convergents, the partial numerators, and denominators, as well as keep in mind the three-term recurrence relations.

We define the sequence $\{r_n\}_{n\geqslant 0}$ by

$$r_0=1,\quad r_1=\frac{1}{\bar a_0 z},\quad r_2=\bar a_0 z,\quad r_3=\frac{1}{\bar a_1 z},\ \ldots,\quad r_{2n}=\bar a_{n-1}z,\quad r_{2n+1}=\frac{1}{\bar a_n z},\ \ldots,$$

which in view of (7.2) results in the following continued fraction

$$a_0+\frac{(1-|a_0|^2)\,\dfrac{1}{\bar a_0}}{1}\ \genfrac{}{}{0pt}{}{}{+}\ \frac{1}{a_1\bar a_0 z}\ \genfrac{}{}{0pt}{}{}{+}\ \frac{(1-|a_1|^2)\,\dfrac{\bar a_0}{\bar a_1}\,z}{1}\ \genfrac{}{}{0pt}{}{}{+\cdots} \tag{7.4}$$

$$\genfrac{}{}{0pt}{}{}{+}\ \frac{1}{a_n\bar a_{n-1}z}\ \genfrac{}{}{0pt}{}{}{+}\ \frac{(1-|a_n|^2)\,\dfrac{\bar a_{n-1}}{\bar a_n}\,z}{1}\ \genfrac{}{}{0pt}{}{}{+\cdots}\ .$$

The fact that (7.4) is a limit-periodic continued fraction of order 2 (see [38, Band II, Chap. II, Sect. 19,II] for a definition) follows from the López condition, since

$$\lim_n\, a_n\bar a_{n-1}z=\lim_n\,\frac{a_n}{a_{n-1}}\,\lim_n\,|a_{n-1}|^2\,z=a^2\lambda z,$$

$$\lim_n\,(1-|a_n|^2)\,\frac{\bar a_{n-1}}{\bar a_n}\,z=\lim_n\,(1-|a_n|^2)\,\lim_n\,\frac{\bar a_{n-1}}{\bar a_n}\,z=(1-a^2)\,\lambda z.$$

In 1952, Perron [11, Theorem 4.4] observed that the "limit" continued fraction for. (7.4))

$$a+\frac{(1-a^2)\,z}{az}\ \genfrac{}{}{0pt}{}{}{+}\ \frac{1}{a}\ \genfrac{}{}{0pt}{}{}{+}\ \frac{(1-a^2)\,z}{az}\ \genfrac{}{}{0pt}{}{}{+\cdots+}\ \frac{1}{a}\ \genfrac{}{}{0pt}{}{}{+}\ \frac{(1-a^2)\,z}{az}\ \genfrac{}{}{0pt}{}{}{+\cdots}$$

converges uniformly on subsets of $\mathbb{C}\setminus\Delta_\alpha$, where $\sin(\alpha/2)=a$, to the holomorphic function $f_a(z)$ defined by (1.22); i.e.

$$\lim_n\,\frac{A_n(z)}{B_n(z)}=\lim_n\,\frac{B_n^*(z)}{A_n^*(z)}=f_a(z),\qquad z\in\mathbb{C}\setminus\Delta_\alpha. \tag{7.5}$$

Substituting (1.8) into (1.1), i.e., using the formula

$$F_\sigma(z)=-1+\frac{2}{1-zf}$$

and applying known formulae (see [29, (5.5), (5.6)])

$$\Phi_{n+1} = zB_n^* - A_n^*, \qquad \Psi_{n+1} = zB_n^* + A_n^*,$$
$$\Phi_{n+1}^* = B_n - zA_n, \qquad \Psi_{n+1}^* = B_n + zA_n, \tag{7.6}$$

we obtain from (7.3) and(7.4) the conference of the formal development of $F_\sigma(z)$ into a continued fraction:

$$-1+\frac{2}{1-a_0z} - \frac{(1-|a_0|^2)\,\frac{1}{\bar{a}_0}\,z}{1} + \frac{1}{a_1\bar{a}_0z} + \frac{(1-|a|^2)\,\frac{\bar{a}_0}{\bar{a}_1}\,z}{1+\cdots}.$$

$$\begin{array}{cccccc} -1 & 0 \quad 1 & 2 & 3 & 4 & \\ \frac{1}{0}; & \frac{-1}{1};\ \frac{\Psi_1^*}{\Phi_1^*}; & \frac{-z\Psi_1}{z\Phi_1}; & \frac{\Psi_2^*}{\Phi_2^*}; & \frac{-z\Psi_2}{z\Phi_2}; & \dots . \end{array} \tag{7.7}$$

The proof of Theorem F is now an easy application of the theory of limit-periodic continued fractions [38, Band II, Chap. II, Sect. 19]. However, we provide more details for the sake of further generalizations.

Proof of Theorem F. We need the following theorem [38, Band II, Chap. II, Sect. 19, Theorem 2.42].

THEOREM 7.1. *Let $\{p_n\}_{n\geqslant 0}$ and $\{q_n\}_{n\geqslant 0}$ be two sequences of functions on a set E satisfying*

$$\lim_n p_n = p, \qquad \lim_n q_n = q$$

uniformly on E. Suppose also that there exist $\vartheta \in (0, 1)$ and two positive numbers c, C such that

$$c \leqslant |X_1| \leqslant C, \qquad |X_2/X_1| \leqslant \vartheta \tag{7.8}$$

on E, where X_1, X_2 are the roots of the equation

$$X^2 - qX - p = 0. \tag{7.9}$$

Then there exists a positive integer $N = N(\vartheta, c, C)$ such that the continued fraction

$$q_n + \frac{p_{n+1}}{q_{n+1}} + \frac{p_{n+2}}{q_{n+2}} + \frac{p_{n+3}}{q_{n+3}} + \cdots$$

converges uniformly on E for every $n \geqslant N$.

LEMMA 7.2. *Let $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$ for $\sigma\in\mathscr{P}$, and $\{\psi_n\}_{n\geqslant 0}$ the orthogonal polynomials of the second kind for $\{\varphi_n\}_{n\geqslant 0}$. Then*

$$\frac{\psi_n^*}{\varphi_n^*}\rightrightarrows F_\sigma \text{ uniformly in } \mathbb{D}; \qquad (7.10)$$
$$\frac{\psi_n}{\varphi_n}\rightrightarrows -F_\sigma \text{ uniformly in } \{z\colon |z|>1\}.$$

Next,

$$\left|\frac{\psi_n^*(z)}{\varphi_n^*(z)}\right|\leqslant\frac{2}{1-|z|},\quad |z|<1;\qquad \left|\frac{\psi_n(z)}{\varphi_n(z)}\right|\leqslant 1+\frac{2}{|z|-1},\quad |z|>1. \qquad (7.11)$$

Proof. The inequalities (7.11) follow from the well-known formula

$$\frac{\psi_n^*(z)}{\varphi_n^*(z)}=\int_{\mathbb{T}}\frac{\zeta+z}{\zeta-z}\,\frac{dm(\zeta)}{|\varphi_n(\zeta)|^2}, \qquad (7.12)$$

which implies (put $z=0$) that $|\varphi_n|^{-2}\,dm\in\mathscr{P}$. See [17, (1.15)] or [29, (5.10)]. By (7.11) the families of homomorphic functions considered in (7.10) are normal in the corresponding domains. Finally, ψ_n^*/φ_n^* and F_σ heave the same Taylor polynomials of order n at $z=0$ [17, (1.15')]. ▮

Assuming that $\{a_n\}_{n\geqslant 0}$ satisfies (L), we apply Theorem 7.1 to the continued fraction (1.6). It is useful to observe that (1.6) is the odd part of (7.7). It follows from (L) and (1.6) that

$$\lim_n p_n=-(1-a^2)\,\lambda z,\qquad \lim_n q_n=1+\lambda z$$

uniformly on compact subsets of $\mathbb{C}$. Thus (7.9) takes the form

$$X^2-(1+\lambda z)\,X+(1-a^2)\,\lambda z=0$$

and therefore

$$X_1(z)=\tfrac{1}{2}\left\{(1+\lambda z)+\sqrt{(1-\lambda z)^2+4a^2\lambda z}\right\},$$
$$X_2(z)=\tfrac{1}{2}\left\{(1+\lambda z)-\sqrt{(1-\lambda z)^2+4a^2\lambda z}\right\}.$$

We already observed in Section 1 that X_1 maps $\mathbb{C}\setminus\bar{\lambda}\Delta_\alpha$ conformally onto $\{z\colon |z|>\sqrt{1-a^2}\}$. Similarly X_2 maps this domain onto the disk $\{z\colon |z|<\sqrt{1-a^2}\}$. It follows that (7.8) holds on any compact subset E of

$\mathbb{C}\setminus\bar{\lambda}\Delta_\alpha$. By Theorem 7.1 there is a positive integer $N=N(E)$ such that the convergents $\{w_{n,k}\}_{k\geqslant 1}$ of the continued fraction

$$w_n(z)=-\cfrac{(1-|a_{n-1}|^2)\dfrac{a_n}{a_{n-1}}z}{1+\dfrac{a_n}{a_{n-1}}z}+\cfrac{(1-|a_n|^2)\dfrac{a_{n+1}}{a_n}z}{1+\dfrac{a_{n+1}}{a_n}z}-\cdots$$

$$-\cfrac{(1-|a_{n+k-2}|^2)\dfrac{a_{n+k-1}}{a_{n+k-2}}z}{1+\dfrac{a_{n+k-1}}{a_{n+k-2}}z}-\cdots$$

converge to w_n uniformly on E for every $n\geqslant N(E)$. It follows that w_n is homomorphic on E.

The convergents of (1.6), being equal to Ψ_n^*/Φ_n^*, $n=0,1,2,\ldots$, by (7.7), are related with the convergents $\{w_{n,k}\}_{k\geqslant 1}$ by the fundamental relations (see [38, Band II, Chap I, Sect. 1, (5), (T)])

$$\frac{\Psi_{n+k}^*(z)}{\Phi_{n+k}^*(z)}=\frac{\Psi_{n-1}^*(z)\,w_{n,k}+\Psi_n^*(z)}{\Phi_{n-1}^*(z)\,w_{n,k}+\Phi_n^*(z)}. \tag{7.13}$$

Since the function $\Phi_{n-1}^*w_n+\Phi_n^*(z)$ is holomorphic on E, it may have on E only a finite numbers of zeros $e_1,\ldots,e_s$. By Theorem 7.1 the sequence at the right-hand side of (7.13) converges to a holomorphic function uniformly on compact subsets of $E\setminus\{e_1,\ldots,e_s\}$ as $k\to\infty$. On the other hand, the sequence in the left-hand side of (7.13) converges to F_σ uniformly on compact subsets of $\mathbb{D}$, as $k\to\infty$, by Lemma 7.2. Assuming that E is a connected compact with $E\cap\mathbb{D}\neq\varnothing$, we obtain that

$$F_\sigma(z)=\frac{\Psi_{n-1}^*(z)\,w_n(z)+\Psi_n^*(z)}{\Phi_{n-1}^*(z)\,w_n(z)+\Phi_n^*(z)},\qquad z\in E\setminus\{e_1,\ldots,e_s\}, \tag{7.14}$$

for $n\geqslant N(E)$. By the Cauchy–Schwarz formula for generalized functions

$$\bar{\partial}F_\sigma=4\pi iz\sigma, \tag{7.15}$$

applied to (7.14), we obtain that $\sigma(E\setminus\{e_1,\ldots,e_s\})=0$. Hence $(supp\sigma)'\subset\bar{\lambda}\Delta_\alpha$. The opposite inclusion follows from Theorem B.

The following formula

$$\Phi_{n-1}^*(\Psi_{n-1}^*w_n+\Psi_n^*)-\Psi_{n-1}^*(\Phi_{n-1}^*w_n+\Phi_n^*)=2a_{n-1}z^n\prod_{k=0}^{n-2}(1-|a_k|^2),$$

which is a direct consequence of the recurrence relations (see [17, (1.17), (8.1)]), shows that the numerator and denominator in (7.14) cannot have common zeros on E if $0 \notin E$. Since F_σ is obviously holomorphic on $\hat{\mathbb{C}} \setminus \mathbb{T}$, we obtain that the points $e_1, ..., e_s$ must lie on $\mathbb{T}$ and carry the point masses of σ. Finally, if we take E with $E \cap supp(\sigma) = \varnothing$, we obtain by (7.13) that $\Psi_n^*/\Phi_n^* \rightrightarrows F_\sigma$ uniformly on E. The convergence $\Psi_n/\Phi_n \rightrightarrows -F_\sigma$ follows by complex conjugation. ▮

8. NORMAL FAMILIES AND ORTHOGONAL POLYNOMIALS

8.1. The proof of Theorem F given in Section 7 is based on two important observations. First, the López condition combined with an algebraic construction (i.e., a continued fraction) implies the normality of the convergents ψ_n/φ_n in the corresponding domain. Next, (L) implies $supp(\sigma) \neq \mathbb{T}$.

We are going now to consider a more general case, in which (L) is replaced with $supp(\sigma) \neq \mathbb{T}$. Following the proof of Theorem F we study normal families of special holomorphic functions. It should be noticed that normal families of similar functions play an important rôle in the convergence theory of continued fractions; see [25].

Definition 8.1. A family $\mathfrak{F}$ of holomorphic functions on an open subset $\mathcal{O}$ of $\hat{\mathbb{C}}$ is called *normal* if $\mathfrak{F}$ is a relatively compact subset of the space $\mathscr{A}(\mathcal{O})$ of all holomorphic functions on $\mathcal{O}$ equipped with the topology of uniform convergence on compact subsets of $\mathcal{O}$.

A family $\mathfrak{F}$ of functions holomorphic at the points of a compact set $K \subset \mathbb{C}$ is normal (on K) if there is an open set $\mathcal{O}$ such that $\mathfrak{F} \mid \mathcal{O} \subset \mathscr{A}(\mathcal{O})$ and $\mathfrak{F} \mid \mathcal{O}$ is normal on $\mathcal{O}$.

In 1938, Levinson obtained a useful general criterion for a family of holomorphic functions to be normal.

Theorem (Levinson [30, Chap. VIII, Theorem XLIII, p. 127]). *Let $\mathscr{M}$ be a positive decreasing function on* $(0, b)$ *with* $\lim_{y \to 0+} \mathscr{M}(y) = +\infty$. *Let $\mathfrak{F}_{\mathscr{M}}$ be a family of functions holomorphic at the points of the rectangle*

$$\Delta = \{z \in \mathbb{C}: |x| \leqslant a, |y| \leqslant b, z = x + iy\},$$

where $a > 0$, *and satisfying*

$$|f(z)| < \mathscr{M}(\Im z) = \mathscr{M}(y) \qquad z \in \Delta.$$

Suppose that

$$\int_0^b \log\log(\mathscr{M}(y)+1)\,dy < +\infty.$$

Then for every $\delta > 0$ *there is a positive constant* $C(\delta)$ *such that for every* $f \in \mathfrak{F}_{\mathscr{M}}$ *we have*

$$|f(z)| < C(\delta), \qquad |x| < a-\delta, \qquad |y| < b, \qquad z = x+iy.$$

We refer the interested reader, to [22] for a simple proof as well as for a discussion of further results, such as presented in [7] and [8]. In fact we only use Levinson's theorem for the special case $\mathscr{M}(y) = 1/y$, when the proof is easy.

Proof of the special case. It is clear that the family of holomorphic functions

$$(z^2-a^2)\,f(z), \qquad f \in \mathfrak{F}_{\mathscr{M}},$$

is uniformly bounded on the boundary of Δ by some positive constant C_0. Since these functions are holomorphic on Δ, by the maximum modulus theorem we obtain that $|(z^2-a^2)\,f(s)| \leqslant C_0$, $z \in \Delta$, which implies the desired estimate with $C(\delta) = (a\delta)^{-1}\,C_0$. ∎

In what follows we apply Levinson's theorem to circular arcs rather than to segments of the real line and to closed disk Δ with the boundary orthogonal to $\mathbb{T}$ rather than to the rectangles described in Levinson's theorem. It is easy to see that the above proof works in this situation as well. For brevity we refer to such disks Δ as *orthogonal* disks. It is important to observe that the family of all open orthogonal disks containing a given point t, $t \in \mathbb{T}$, is the base of the Euclidean topology at t. Notice also that Schwarz's symmetry $z \to 1/\bar{z}$, keeping the circle $\mathbb{T}$ invariant, maps any orthogonal disk onto itself, see [9, Chap. V, Sect. 2]. Given a subset E of the closed disk $\{z\colon |z| \leqslant 1\}$ we define E^* to be the image of E under Schwarz's mapping $z \to 1/\bar{z}$. If E is compact and $0 \notin E$, then, obviously, E^* is compact.

The following definition is a convenient technical tool to distinguish two cases in the proof of Theorem G, (3).

Definition 8.2. Let $\sigma \in \mathscr{P}$. A point $t \in supp(\sigma)$ is called analytic if there is an open arc I centered at t such that the restriction of the measure σ to I is an absolutely continuous measure with analytic density on I. The set of all analytic points of σ is denoted by $an(\sigma)$.

It is clear that $an(\sigma)$ is an open subset of $\mathbb{T}$.

THEOREM 8.3. *Let $\sigma \in \mathscr{P}$, $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$ and $\{\psi_n\}_{n\geqslant 0}$ the associated polynomials of the second kind. Let K, $0 \notin K$, be a connected compact subset of the closed disk $\{z: |z| \leqslant 1\}$ such that there exists a positive constant C, an infinite subset $\Lambda \subset \mathbb{Z}_+$, a closed orthogonal disk $\bar{\Delta}$, $\bar{\Delta} \cap \mathbb{D} \subset K$, and an open set $\mathscr{O}$, $K \subset \mathscr{O}$, satisfying*

$$\left|\frac{\psi_n(z)}{\varphi_n(z)}\right| \leqslant \frac{C}{1-|z|}, \qquad z \in \mathscr{O} \cap \mathbb{D}, \tag{8.1}$$

$$\frac{\psi_n}{\varphi_n} \in \mathscr{A}(\mathscr{O}), \qquad n \in \Lambda. \tag{8.2}$$

Then either $\bar{\Delta} \cap \mathbb{T} \subset an(\sigma)$ or $\sigma(\bar{\Delta} \cap \mathbb{T}) = 0$ and

$$\lim_n \frac{\psi_n(z)}{\varphi_n(z)} = \int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta}\, d\sigma(\zeta) \tag{8.3}$$

uniformly on $K \cup K^*$.

Proof. Let Δ_1 be a larger open orthogonal disk such that $\bar{\Delta} \subset \Delta_1$ and $\bar{\Delta}_1 \cap \mathbb{D} \subset \mathscr{O}$. By (8.1) and (7.11) the family $\{\psi_n/\varphi_n\}_{n\in\Lambda}$ satisfies the assumptions of Levinson's theorem for the orthogonal disk $\bar{\Delta}_1$. It follows that the family $\{\psi_n/\varphi_n\}_{n\in\Lambda}$ is normal on $\bar{\Delta}$. Since Δ is an orthogonal disk, the sequence $\{\psi_n/\varphi_n\}_{n\geqslant 0}$ converges to $-F_\sigma$ uniformly on compact subsets of the nonempty open set $\Delta \cap \{z: |z| > 1\}$, as $n \to \infty$, by (7.10) of Lemma 7.2. It follows that

$$\frac{\psi_n(z)}{\varphi_n(z)} \underset{n\in\Lambda}{\rightrightarrows} F_1(z) \tag{8.4}$$

uniformly on $\bar{\Delta}$. Since Δ is invariant under Schwarz's symmetry,

$$\frac{\psi_n^*(z)}{\varphi_n^*(z)} \underset{n\in\Lambda}{\rightrightarrows} F_2(z) \tag{8.5}$$

uniformly on $\bar{\Delta}$. By Lemma 7.2 we have $F_2(z) = F_\sigma(z)$ in $\Delta \cap \mathbb{D}$ and is homomorphic on $\bar{\Delta}$. It is well known (see [17] or [29, (5.9)]) that

$$\frac{\psi_n^*}{\varphi_n^*} + \frac{\psi_n}{\varphi_n} = \frac{2z^2}{\varphi_n^*\varphi_n} = \frac{2}{|\varphi_n|^2} \tag{8.6}$$

on the unit circle $\mathbb{T}$, and that (see [40] or [29, Lemma 5.4])

$$(*)-\lim_n \frac{dm}{|\varphi_n|^2} = d\sigma \tag{8.7}$$

in the $(*)$-weak topology of $M(\mathbb{T})$. Let $J=\bar{\Delta}\cap\mathbb{T}$. Combining (8.7) with (8.4), (8.5), and (8.6), we obtain that the restriction of σ to J is absolutely continuous and its density equals the restriction to J of the holomorphic function $F=F_1+F_2$. Now either $F\equiv 0$ or $F\not\equiv 0$.

In the first case by the uniqueness theorem $F\equiv 0$ in Δ. By (7.10) this yields that (8.3) holds uniformly on Δ. Therefore $F_\sigma(z)$ is holomorphic on Δ, and hence $\sigma(\bar{\Delta})=0$ by (7.15). This in turn implies that $\bar{\Delta}\cap supp(\sigma)=\varnothing$.

If $F\not\equiv 0$, then by the definition, $\bar{\Delta}\subset an(\sigma)$ as stated. ▮

8.2. *Extensions of Theorem* F. Our plan is to extend Theorem F to probability measures σ with $supp(\sigma)\neq\mathbb{T}$. We need a substitute of Perron's theorem (Theorem 7.1) to control the behavior of the zeros of the polynomials orthogonal in $L^2(\sigma)$, i.e., of the zeros of the inverse Schur functions of σ. Recall that any measure σ with $supp(\sigma)\neq\mathbb{T}$ is in $Mar(\mathbb{T})$ (see Sect. 1).

LEMMA 8.4. *Let* $\sigma\in \mathrm{Mar}(\mathbb{T})$. *Then*

(1) *there exists a positive number* $\delta(\sigma)$ *such that*

$$\sup_{|z|\leqslant 1/2}|b_n(z)|\geqslant\delta(\sigma)>0,\qquad n=1,2,\ldots;\tag{8.8}$$

(2) *Every* $g\in\mathscr{B}(\sigma)$ *is unimodular, holomorphic on* $\mathbb{T}\setminus(supp(\sigma))'$, *and extends to a meromorphic function in* $\hat{\mathbb{C}}\setminus(supp(\sigma))'$ *by Schwarz's reflection principle;*

(3) *for every* $g\in\mathscr{B}(\sigma)$ *the equation* $zg=1$ *has no solutions on* $\mathbb{T}\setminus(supp(\sigma))$.

Proof. It is clear that the formula

$$\rho(f,g)=sup_{|z|<1/2}|f(z)-g(z)|$$

defines on $\mathscr{B}$ a metric compatible with the $(*)$-weak topology. By Theorem 1.8(3) $0\notin\mathscr{B}_0(\sigma)$, which yields

$$\delta_0(\sigma)=\inf\{\rho(f,0)\colon f\in\mathscr{B}_0(\sigma)\}>0.$$

Since the set $\mathscr{B}_0(\sigma)\cup\{\frac{-a_{n-1}}{|a_{n-1}|}b_n:=1,2,\ldots\}$ is compact and the function $x\to\rho(x,0)$ is continuous and positive on this set, we obtain (8.7).

To prove (2) assume that $supp(\sigma)\neq\mathbb{T}$. It is well known that

$$\sum_{k=0}^{\infty}|\varphi_k(t)|^2=\frac{1}{\sigma(t)},\qquad t\in\mathbb{T},\tag{8.9}$$

See [16, Sect. 20] for a proof or [17, Remarks to Chap. I, 1.1] for a proof of the inequality $\leqslant$, which is all we need in what follows. Let I be a closed arc on $\mathbb{T}$ such that $I \cap (supp(\sigma))' = \varnothing$. Then $Card\{t \in I : \sigma\{t\} > 0\}$. By (8.8) we have

$$\lim_n \int_I |\varphi_n|^2 \, d\sigma = 0,$$

which implies that $\nu(\mathbb{T} \setminus (supp(\sigma))') = 0$ for every $\nu \in \operatorname{Lim}(\sigma)$. It follows from Theorem 1.4 that

$$F_\nu(z) = \int_{(supp(\sigma))'} \frac{\zeta + z}{\zeta - z} \, d\nu(\zeta) = \frac{1 + zg_\nu}{1 - zg_\nu}; \qquad zg_\nu = \frac{F_\nu - 1}{F_\nu + 1}, \tag{8.10}$$

where g_ν is the Schur function of ν. Clearly, $F_\nu(z)$ is holomorphic on $\hat{\mathbb{C}} \setminus (supp(\sigma))'$. It is also clear that $\Re F_\nu = 0$ on $\mathbb{T} \setminus (supp(\sigma))'$. It follows that $|g| = 1$ on $\mathbb{T} \setminus (supp(\sigma))'$, which implies (2).

The statement (3) follows from the observation that $F_\nu(z)$ cannot assume infinite values on $\mathbb{T} \setminus (supp(\sigma)')$, since it is holomorphic on this set. ▮

An elementary proof of the following lemma can be found in [13, Chap. VII, Lemma 1].

Lemma 8.5. *Let $B(z)$ be the Blaschke product in $\mathbb{D}$ with zeros $\{z_j\}$. Then*

$$-\log |B(z)|^2 \geqslant (1 - |z|^2) \sum_j \frac{1 - |z_j|^2}{|1 - \bar{z}_j z|^2}, \qquad z \in \mathbb{D}. \tag{8.11}$$

Conversely, if

$$\inf_j \left| \frac{z - z_j}{1 - \bar{z}_j z} \right| \geqslant a > 0,$$

then

$$-\log |B(z)|^2 \leqslant \left(1 + \log \frac{1}{a^2} \right) (1 - |z|^2) \sum_j \frac{1 - |z_j|^2}{|1 - \bar{z}_j z|^2}. \tag{8.12}$$

Corollary 8.6. *Let B be a Blaschke product with zeros $\{z_j\}$ such that $|B(z_0)| > \delta > 0$ for some z_0. $|z_0| \leqslant 1/2$. Then*

$$\sum_{B(z_j) = 0} (1 - |z_j|^2) \leqslant 3 \log \frac{1}{\delta^2}. \tag{8.13}$$

Next, if $|z-z_j| \geqslant a_0 > 0$, *then*

$$|B(z)| > c = c(\delta, a_0) > 0. \tag{8.14}$$

Proof. Inequality (8.13) follows from (8.10) and the following obvious estimates

$$\sum_j (1-|z_j|^2) \leqslant 3\,\frac{1-|z_0|}{1+|z_0|} \sum_j (1-|z_j|^2) \leqslant 3(1-|z_0|^2) \sum_j \frac{1-|z_j|^2}{|1-\bar{z}_j z_0|^2} \leqslant 3 \log \frac{1}{\delta^2}.$$

To prove (8.13) we observe first that the inequality $|z-z_j| \geqslant a_0$ implies

$$\left|\frac{z-z_j}{1-\bar{z}_j z}\right| \geqslant \frac{a_0}{|1-\bar{z}_j z|} \geqslant \frac{a_0}{2}.$$

Hence (8.11) holds with $a = a_0/2$ and we obtain by (8.11) and (8.12) that

$$\begin{aligned}
\log \frac{1}{|B(z)|^2} &\leqslant \left(1+\log \frac{4}{a_0^2}\right) \sum_j \frac{1-|z_j|^2}{|z-z_j|^2} \left|\frac{z-z_j}{1-\bar{z}_j z}\right| \\
&\leqslant \left(1+\log \frac{4}{a_0^2}\right) \sum_j \frac{1-|z_j|^2}{|z-z_j|^2} \leqslant \frac{1}{a_0^2}\left(1+\log \frac{4}{a_0^2}\right) \sum_j (1-|z_j|^2) \\
&\leqslant 3\,\frac{1}{a_0^2}\left(1+\log \frac{4}{a_0^2}\right) \log \frac{1}{\delta^2}.
\end{aligned}$$

If we put now

$$c(\delta, a_0) = \exp\left\{-\frac{3}{a_0^2} \log\left(\frac{4e}{a_0^2}\right) \log \frac{1}{\delta}\right\},$$

then we obtain (8.13). ∎

LEMMA 8.7. *Let* $\sigma \in \mathrm{Mar}(\mathbb{T})$, *and* $\{\varphi_n\}$ *be the orthogonal polynomials in* $L^2(d\sigma)$. *Then for every orthogonal disk* $\Delta \cap \mathbb{T} \subset an(\sigma)$ *there is a positive integer* $N(\Delta)$ *such that for any integer* $n > N(A)$ *the orthogonal polynomial* φ_n *has at least one zero in* Δ.

Proof. Assuming the contrary, we obtain an infinite sequence Λ such that no polynomial φ_n may vanish at Δ if $n \in \Lambda$. Let b_n be the inverse Schur functions of σ. Since $\sigma \in \mathrm{Mar}(\mathbb{T})$, we obtain by Lemma 8.4 that the inverse Schur iterates for σ satisfy (8.8). In addition our assumption implies that b_n, $n \in \Lambda$, do not have zeros in Δ. Let Δ_0 be a smaller orthogonal disk in Δ such that its distance to the boundary of Δ is greater than $a_0 > 0$. Then by

Corollary 8.6 every Blaschke product b_n, $n \in \Lambda$ satisfies (8.14). It follows that

$$\inf_{n \in \Lambda} \inf_{z \in \Delta_0 \cap \mathbb{D}} |b_n(z)| > c > 0. \tag{8.15}$$

Let λ_n be the nth inverse Schur iterates for the associated orthogonal polynomial ψ_n of the second kind; i.e., $\lambda_n = \psi_n/\psi_n^*$. Then (8.15) and the elementary identity

$$\frac{\psi_n}{\varphi_n} = \frac{\lambda_n}{b_n} \frac{\psi_n^*}{\varphi_n^*}$$

imply that

$$\left|\frac{\psi_n(z)}{\varphi_n(z)}\right| \leqslant \frac{1}{c} \left|\frac{\psi_n^*(z)}{\varphi_n^*(z)}\right| \leqslant \frac{2}{c} \frac{1}{1-|z|} \tag{8.16}$$

for $z \in \Delta_0 \cap \mathbb{D}$; see (7.11). It follows that the family $\{\psi_n/\varphi_n\}_{n \in \Lambda}$ is normal in a smaller orthogonal disk Δ_1. It follows that

$$\lim_{n \in \Lambda} \frac{\psi_n}{\varphi_n} = F_1 \tag{8.17}$$

uniformly on $J_1 = \Delta_1 \cap \mathbb{T}$. To complete the proof we need a simple modification of one important lemma from [29]. We provide a proof for the reader's convenience.

LEMMA 8.8 (compare with [29, Lemma 8.2]). *Let σ be a probability measure such that $\sigma' > 0$ on a set E, $m(E) > 0$. Let A_n, B_n be the Wall polynomials corresponding to the Schur function f of σ and Λ be an infinite subset of $\mathbb{Z}_+$. Suppose that the sequence $\{A_n/B_n\}_{n \in \Lambda}$ converges in measure (with respect to m) to some function X on E, as $n \underset{n \in \Lambda}{\to} \infty$. Then $0 \in \mathscr{B}(\sigma)$.*

Proof. First, we observe that $X = f$ a.e. on E. Indeed, by Wall's theorem [48], [29, Corollary 4.7] we have $(*)-\lim_n A_n/B_n = f$ in the $(*)$-weak topology of $\mathscr{B}$. Hence

$$\lim_n \int_E \frac{A_n}{B_n} h \, dm = \int_E hf \, dm \tag{8.18}$$

for every h, $h \in L^1(\mathbb{T})$. Applying Lebesgue's dominated convergence theorem to the subsequence $\{A_n/B_n\}_{n \in \Lambda}$, we conclude that the limit in the left-hand side of (8.18) is nothing but

$$\int_E Xh \, dm = \int_E hf \, dm.$$

Since h is an arbitrary function in $L^1(\mathbb{T})$, this yields $X = f$ a.e., on E as stated.

Next, we apply a simple identity (see [29, (4.19)]):

$$\frac{A_{n+k}}{B_{n+k}} = \frac{A_n + zB_n^* f_{n+1,\,n+k}}{B_n + zA_n^* f_{n+1,\,n+k}}. \tag{8.19}$$

Here we denote by $f_{n+1,\,n+k}$ the rational function with the parameters

$$(a_{n+1}, a_{n+2}, \ldots, a_{n+k}, 0, \ldots).$$

It is easy to see that

$$\frac{A_{n+k}}{B_{n+k}} - \frac{A_n}{B_n} = \frac{\omega_n z^{n+1}}{B_n{}^2} \frac{f_{n+1,\,n+k}}{1 + z\dfrac{A_n^*}{B_n} f_{n+1,\,n+k}}. \tag{8.20}$$

Notice that the modulus of the first fraction in the right-hand side of (8.20) is

$$\frac{\omega_n}{|B_n|^2} = 1 - |A_n/B_n|^2$$

which converges in measure on E, as $n \to +\infty$ along Λ, to some positive function, in fact, to $1 - |f|^2$, as it has already been shown. Next, by this very reason the denominator of the second fraction in (8.20) is bounded away from zero on E at least on a subset of positive Lebesgue measure. So, if n and $n+k$ are elements of Λ, then the left-hand side of (8.20) tends to zero as $n \to \infty$ along Λ and $n+k \in \Lambda$. This implies that

$$f_{n,\,n+k} \rightrightarrows 0 \tag{8.21}$$

on a subset of positive Lebesgue measure of E, when $n, n+k \in \Lambda$ tend to ∞.

Now we may apply the Khinchin–A. Ostrowski theorem [39, Chap. 2, Sect. 7].

THEOREM (Khinchin–A. Ostrowski). *Let $\{h_n\}_{n \geqslant 0}$ be a sequence of functions in $\mathscr{B}$ such that their boundary values on the unit circle converge in measure, $h_n \rightrightarrows h$, on a set $E \subset \mathbb{T}$ of positive Lebesgue measure $m(E) > 0$. Then there is a unique function $\hat{h} \in \mathscr{B}$ such that $h_n \to \hat{h}$ in $\mathscr{B}$, i.e., uniformly on compact subsets of $\mathbb{D}$, and $\hat{h} = h$ a.e. on E.*

By the Khinchin–Ostrowsky theorem

$$\lim_{n,\,n+k \in \Lambda} f_{n,\,n+k} = 0 \tag{8.22}$$

in $\mathscr{B}$. By Theorem 1.1 this implies that the sequence of Schur's parameters $\{a_n\}_{n\geqslant 0}$ contains arbitrary long segments of arbitrary small parameters. It follows that condition (1.28) cannot hold and hence $0 \in \mathscr{B}(\sigma)$ by Theorem 1.8. ▮

We can now complete the proof of Lemma 8.7. By (7.6) we have

$$\frac{\psi_{n+1}}{\varphi_{n+1}} = \frac{\Psi_{n+1}}{\Phi_{n+1}} = \frac{zB_n^* + A_n^*}{zB_n^* - A_n^*} = \frac{z + \dfrac{A_n^*}{B_n^*}}{z - \dfrac{A_n^*}{B_n^*}},$$

which implies the pointwise convergence of $\{A_n/B_n\}$ along an infinite sequence on the arc J_1. By Lemma 8.8 we obtain a contradiction. ▮

8.4. *Proof of Theorem* G.

Proof of Theorem G. (1) By Lemma 8.4(1) the inverse Schur iterates for σ satisfy (8.8). Hence by Corollary 8.6 every b_n satisfies (8.13) with $\delta = \delta(\sigma)$ independent of n.

(2) It follows from the obvious inequality

$$(1-r^2)\, n(r) \leqslant \sum_{k=1}^{n} (1-|\lambda_{kn}|^2) \leqslant 6 \log \frac{1}{\delta(\sigma)}.$$

(3) Suppose first that $t \in supp(\sigma) \setminus an(\sigma)$ and assume to the contrary that there are an open orthogonal disk Δ containing $t \in supp(\sigma) \setminus an(\sigma)$ and an infinite subset $\Lambda \subset \mathbb{Z}_+$ such that no polynomial φ_n, $n \in \Lambda$ vanishes on Δ. By Lemma 8.4(1) the functions b_n satisfy (8.8). Hence by Corollary 8.6 the Blaschke products b_n, $n \in \Lambda$ satisfy (8.14) in a smaller orthogonal disk $\Delta_0 \subset \Delta$, which also contains t. By (8.14) the family $\{\psi_n/\varphi_n\}_{n\in\Lambda}$ satisfies the assumptions of Theorem 8.3. Since $t \notin an(\sigma)$ by the assumption, we conclude from Theorem 8.3 that $\sigma(\Delta_0 \cap \mathbb{T}) = 0$, which obviously contradicts the other assumption that $t \in supp(\sigma)$.

The case $t \in an(\sigma)$ follows by Lemma 8.7.

(4) This part of Theorem G is an immediate consequence of the well-known fact that the zeros of orthogonal polynomials are contained in the convex hull of the support of the measure (see the beginning of Section 9 for a simple proof). ▮

9. THE CONVERGENCE OF WALL'S CONTINUED FRACTIONS

9.1. It is well known that the zeros of any orthogonal polynomial φ_n lie in the convex hull $conv(\sigma)$ of $supp(\sigma)$. This is a general fact, which holds for any probability measure with compact support on $\mathbb{C}$. The following simple proof will be used later in the convergence problem of Wall continued fractions.

Proof. Setting $\varphi_n = (z-\lambda)\, p_{n-1}$, we obtain by $\varphi_n \perp p_{n-1}$ that

$$0 = \int_{\mathbb{T}} \varphi_n \bar{p}_{n-1}\, d\sigma = \int_{\mathbb{T}} (z-\lambda)\, |p_{n-1}|^2\, d\sigma, \tag{9.1}$$

which implies that

$$\lambda = \int_{supp(\sigma)} \zeta\, d\mu_n, \qquad d\mu_n = \left(\int_{\mathbb{T}} |p_{n-1}|^2\, d\sigma \right)^{-1} \frac{|\varphi_n|^2\, d\sigma}{|z-\lambda|^2}, \tag{9.2}$$

μ_n being a probability measure supported by $supp(\sigma)$. ∎

LEMMA 9.1. *Let $\sigma \in \mathscr{P}$, $supp(\sigma) \neq \mathbb{T}$. Then*

$$\lim_n \frac{\psi_n(z)}{\varphi_n(z)} = \int_{\mathbb{T}} \frac{z-\zeta}{z-\zeta}\, d\sigma(\zeta) \tag{9.3}$$

uniformly on compact subsets of $\mathbb{C} \setminus conv(\sigma)$.

Proof. The open set $\mathbb{T} \setminus supp(\sigma)$ is the union of at most a countable family $\{I_j\}$ of disjoint open arcs. Then $conv(\sigma)$ is the compact set whose boundary consists of the points of $supp(\sigma)$ and of the points of the chords connecting the end-points of the arcs I_j. By (7.10) the formula (8.11) holds on compact subsets of $\{|z|>1\}$. Since any compact subset of $\mathbb{C} \setminus conv(\sigma)$ is covered by the union of a compact subset of $\{z: |z|>1\}$ with a finite number of closed orthogonal disk $\bar{\Delta}$, $\bar{\Delta} \cap conv(\sigma) = \varnothing$, by (7.10) it is sufficient to prove (9.3) for $z \in \bar{\Delta}$, $\bar{\Delta}$ being the closed orthogonal disk.

Let us fix such a disk. Since σ is a Markoff measure, the inverse Schur iterates b_n of σ satisfy (8.8). By (9.1) the zeros of φ_n are located in $conv(\sigma)$. Since the latter set is separated from $\bar{\Delta}$, b_n's satisfy (8.14) by Corollary 8.6. Hence by (8.16) the family $\{\psi_n/\varphi_n\}$ is normal (possibly on a smaller orthogonal disk). Since $\sigma(\Delta) = 0$, by Theorem 8.2 we obtain that (8.3) holds uniformly on an orthogonal disk. ∎

Simple and general observations on the behavior of orthogonal polynomials on the point spectrum of probability measure lead to an extension of the convergence domain in (9.3). Given a probability measure σ on $\mathbb{T}$ we denote by $conv_0(\sigma)$ the convex hull of the derived set $(supp(\sigma))'$ of $supp(\sigma)$ and by $supp_p(\sigma)$ the point spectrum of σ, i.e., the set $\{t \in \mathbb{T}: \sigma\{t\} > 0\}$.

Let $\{a_n\}_{n \geqslant 0}$ be the Geronimus parameters of σ. By Favard's theorem [10; 29, Sect. 1] there is a unique probability measure σ_{-1} corresponding to the parameters $\{-a_n\}_{n\geqslant 0}$. It is clear that $\{\varphi_n\}_{n\geqslant 0}$ are the polynomials of the second kind for the polynomials $\{\psi_n\}_{n\geqslant 0}$ orthogonal in $L^2(d\sigma_{-1})$.

THEOREM 9.2. *Let $\sigma \in \mathscr{P}$ with $supp(\sigma) \neq \mathbb{T}$. Then*

$$\lim_n \frac{\psi_n(z)}{\varphi_n(z)} = \int_{\mathbb{T}} \frac{z-\zeta}{z-\bar{\zeta}}\, d\sigma(\zeta) \tag{9.4}$$

uniformly on compact subsets of $\mathbb{C}\setminus(conv_0(\sigma)\cup supp_p(\sigma))$. For any $t\in supp_p(\sigma)$ there exist an open disk Δ centered at t and a positive integer $N(\Delta)$ such that φ_n has exactly one zero in Δ if $n > N(\Delta)$.

Proof. It is based on Lemma 10 of [2], which says that

$$supp_p(\sigma)\cap supp_p(\sigma_{-1}) = \varnothing. \tag{9.5}$$

Indeed, if $\sigma(\{t\})\,\sigma_{-1}(\{t\}) > 0$, then $\lim_n \varphi_n(t) = \lim_n \psi_n(t) = 0$ by (8.9). This, however, contradicts the "determinantal identity"

$$\varphi_n(t)\,\bar{\psi}_n(t) + \bar{\varphi}_n(t)\,\psi_n(t) \equiv 1, \qquad t\in\mathbb{T}; \tag{9.6}$$

see [17, (1.17)]. It follows from $F_{\sigma_{-1}} = F_\sigma^{-1}$, that $supp(\sigma_{-1}) \neq \mathbb{T}$. Hence by Lemma 9.1

$$\lim_n \frac{\varphi_n(z)}{\psi_n(z)} = -\frac{1}{F_\sigma(z)} \tag{9.7}$$

uniformly on compact subsets of $\mathbb{C}\setminus conv(\sigma_{-1})$. Let $t\in supp_p(\sigma)$. Then by (9.5) there exists a disk Δ centered at t such that $\bar{\Delta}\cap conv(\sigma_{-1}) = \varnothing$. It follows that (9.7) holds uniformly on Δ. Since F_σ has a simple pole at t, the function F_σ^{-1} has a simple zero at t. Since $t\in supp_p(\sigma)$, it is clear that $t\notin an(\sigma)$. Hence by Theorem G (3) there is a positive integer $N(\Delta)$ such that φ_n has at least one zero in Δ for $n > N(\Delta)$. Applying Hurwitz's theorem to (8.14) and increasing $N(\delta)$ if necessary, we obtain that φ_n has *exactly one* zero in Δ, $n > N(\Delta)$. By Corollary 1.11 this zero approaches t if $n\to\infty$.

Now let K be a compact subset of $\mathbb{C}\setminus(conv_0(\sigma)\cup supp_p(\sigma))$. Then there exists a finite subset $\{t_1, \ldots, t_n\}$ of $supp_p(\sigma)$ such that the convex hull $\mathscr{F}$ of $supp(\sigma)\setminus\{t_1, \ldots, t_s\}$ does not intersect K. Let N be a positive integer such that all the zeros of φ_n, accumulating to $t_1, \ldots, t_s$, are at a positive distance from K for $n > N$. Let us call these zeros of φ_n *marked.* If λ is not a marked zero of φ_n, then (9.2) shows that the portion of the probability measure μ_n in (9.2), corresponding to the finite set $\{t_1, \ldots, t_s\}$, diminishes as $n\to\infty$. It follows that λ approaches the closed convex set $\mathscr{F}$, as $n\to\infty$. Hence there is a positive N such that all the zeros of φ_n, $n > N$, are at a distance $\geqslant \varepsilon$, $\varepsilon > 0$, from K. This implies that the family $\{\psi_n/\varphi_n\}_{n>N}$ is normal on K. ▌

Remark. Using (7.7), we conclude by Theorem 9.2 that the Wall continued fraction, corresponding to σ, converges on complementary arcs of $supp(\sigma)$ as well as in some neighborhood of any such arc. This is an extension of a well-known Geronimus theorem [14].

9.2. *The set* $Z(\sigma)$. The arguments we used in the proof of Theorems 2.3 and 2.10 show that as soon as the set of accumulation points of the zeros of orthogonal polynomials can be diminished, the convergence domain of ψ_n/φ_n can be enlarged. To be more accurate let us state the following definition.

DEFINITION 9.3. A finite set $Z \subset \mathbb{D}$ is said to be a set of accumulation points for the zeros of $\{\varphi_n\}_{n\geqslant 0}$ if there is an infinite subset Λ of $\mathbb{Z}_+$ such that for every $\varepsilon > 0$ there is a positive integer $N(\varepsilon)$ such that any disk Δ of radius ε centered at any point of Z contains at least one zero of φ_n for $n > N(\varepsilon)$, $n \in \Lambda$.

Observe that by Theorem G (2) the number $n(r)$ of zeros of φ_n in $\{z: |z| \leqslant r\}$ is uniformly bounded. However, this fact alone, obviously, is not enough to control the accumulation process, since the zeros may split into groups corresponding to different subsets Λ of $\mathbb{Z}_+$and accumulate to all finite subsets of $\{z: |z| \leqslant r\}$ subjected only to the restriction of the number of points. To control the accumulation process we use the set $\mathscr{B}_0(\sigma)$ (see Sect. 1).

LEMMA 9.4. *Let σ be a Markoff measure and $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Then a finite set Z in $\mathbb{D}$ is a set of accumulation points for the zeros of $\{\varphi_n\}_{n\geqslant 0}$ if and only if there is a function $g \in \mathscr{B}_0(\sigma)$ satisfying $g \mid Z \equiv 0$.*

Proof. If Z is a set of accumulation points, then passing to a subset of a set Λ from the definition, we may assume that the limit $g = \lim_{n\in\Lambda} b_n$ exists uniformly on compact subsets of $\mathbb{D}$. Since every open disk centered at any point of Z contains a zero of b_n for $n \in \Lambda$, $n > N$, we obtain that $g \mid Z \equiv 0$.

Suppose now that there is $g \in \mathscr{B}_0(\sigma)$ with $g \mid Z \equiv 0$. Since $\sigma \in \mathrm{Mar}(\mathbb{T})$, the function g cannot vanish identically in $\mathbb{D}$. It follows that there is $r \in (0, 1)$ with $Z \subset \{z: |z| \leqslant r\}$ and $\inf_{\{|z|=r\}} |g(z)| > 0$. Since $g \in \mathscr{B}_0(\sigma)$, there is a subset Λ of $\mathbb{Z}_+$ such that $\lim_{n\in\Lambda} b_n = g$ in $\mathscr{B}$. It follows that $\lim_{n\in\Lambda} b_n = g$ uniformly on $\{z: |z| = r\}$. By Hurwitz' theorem the zeros of b_n accumulate to the zeros of g in the disk $\{z: |z| \leqslant r\}$, as $n \to \infty$ along Λ. ▌

For a Markoff measure σ we define the set $Z(\sigma)$ by

$$Z(\sigma) \stackrel{\text{def}}{=} \{z \in \mathbb{D} \colon f(z) = 0, \text{ for some } f \in \mathscr{B}_0(\sigma)\}. \tag{9.8}$$

The following lemma sums up the basic properties of $Z(\sigma)$.

LEMMA 9.5. *If $\sigma \in Mar(\mathbb{T})$, then*

(1) *$Z(\sigma)$ is a closed subset of $\mathbb{D}$ in $\mathbb{D}$;*

(2) *$Z(\sigma) \subset conv_0(\sigma)$.*

Proof. (1) Let z_0 be a point in $\mathbb{D}$ such that $\{z \colon |z - z_0| < 1/n\} \cap Z(\sigma) \neq \varnothing$ for $n = 1, 2, \ldots$. Then for every $n = 1, 2, \ldots$, there is a point z_n, $|z_n - z_0| < 1/n$ and a function $f_n \in \mathscr{B}_0(\sigma)$ such that $f_n(z_n) = 0$. Since $\mathscr{B}_0(\sigma)$ is a compact set bounded away from 0 (by Theorem 1.8), there is an infinite subset $\Lambda \subset \mathbb{Z}_+$ such that $\lim_{n \in \Lambda} f_n = f \not\equiv 0$ in $\mathscr{B}_0(\sigma)$, i.e., uniformly on compact subsets of $\mathbb{D}$. It follows that f is a nonzero function vanishing at z_0. Hence $z_0 \in Z(\sigma)$.

(2) It follows by Lemma 9.4. ▮

By Lemma 9.5(2) the limit points of $Z(\sigma)$ on $\mathbb{T}$ are contained in $(supp(\sigma))'$.

THEOREM 9.6. *Let $\sigma \in \mathscr{P}$ with $supp(\sigma) \neq \mathbb{T}$. Then*

$$\lim_n \frac{\psi_n(z)}{\varphi_n(z)} = \int_{\mathbb{T}} \frac{z + \zeta}{z - \zeta}\, d\sigma(\zeta) \tag{9.9}$$

uniformly on compact subsets of $\mathbb{C} \setminus (supp(\sigma) \cup Z(\sigma))$.

Proof. Let K be a compact subset of the open set $\mathbb{C} \setminus (supp(\sigma) \cup Z(\sigma))$. Then the distance 2ε between K and $supp(\sigma) \cup Z(\sigma)$ is positive. It follows that there is a finite family

$$\Delta_1, \Delta_2, \ldots, \Delta_s \tag{9.10}$$

such that

$$supp(\sigma) \cup Z(\sigma) \subset \bigcup_{j=1}^{s} \Delta_j \tag{9.11}$$

and the disks concentric with Δ_j of twice the radii do not intersect K. Since the family (9.10) is finite and satisfies (9.11), there exists a positive integer $N(K)$ such that for $n \geqslant N(K)$ every zero of φ_n is covered by one of the disks (9.10). Indeed, otherwise we could find an infinite set $\Lambda \subset \mathbb{Z}_+$ such that for every $n \in \Lambda$ there is a zero $\lambda_n = \lambda_{k(n), n}$ of φ_n which is not covered by disks (9.10). Let λ be a limit point of $\{\lambda_n\}_{n\in\Lambda}$. If $\lambda\in\mathbb{D}$, then $\lambda\in Z(\sigma)$ by Lemma 9.5, which contradicts (9.11). If $\lambda\in\mathbb{T}$, then by Lemma 9.1 $\lambda\in supp(\sigma)$, which again contradicts (9.11).

Let us put $\Lambda=\{N(K), N(K)+1, \ldots\}$ in Theorem 8.3 and suppose that $0\notin K$. Then (8.1) and (8.2) hold and the families

$$\left\{\frac{\psi_n}{\varphi_n}\right\}_{n\geqslant N(K)}, \qquad \left\{\frac{\psi_n^*}{\varphi_n^*}\right\}_{n\geqslant N(K)} \tag{9.12}$$

are normal on $K\cup K^*$ by Theorem 8.3.

Suppose first that K is a subset of the connected component $\Omega_\infty(\sigma)$ of $\mathbb{C}\setminus(supp(\sigma)\cup Z(\sigma))$ containing ∞. Then, enlarging K, if necessary, we may assume that K is connected and $K\cap\mathbb{T}\neq\varnothing$. Now the convergence in (2.47) on $K\cup K^*$ follows by Lemma 9.1.

Now let K be a subset of the bounded component of $C\setminus(supp(\sigma)\cup Z(\sigma))$. Since K is compact, it is covered by a finite number of such components. It follows that there exists $r\in(0,1)$ such that

$$K\subset\{z\in\mathbb{D}: |z|<r\}. \tag{9.13}$$

Suppose that the first normal family of (9.12) does not converge to the holomorphic function at the right-hand side of (9.9). Increasing r in (9.13), we may assume that $\{z: |z|<r\}\cap(\mathbb{C}\setminus conv(\sigma))$ is a non-empty open set.

We are going now to arrange a diagonal process which results in a contradiction. Let Λ be an infinite subset of $\mathbb{Z}_+$ such that

$$\lim_{n\in\Lambda}\frac{\psi_n(z)}{\varphi_n(z)}=H(z)$$

uniformly on K, where $H(z)\neq -F_\sigma(z)$ on K. By Theorem G (2) the total number of zeros in $\{z: |z|\leqslant r\}$ for each φ_n is uniformly bounded. Let us somehow enumerate the zeros of each φ_n contained in $\{z: |z|\leqslant r\}$. Then there exists an infinite subset Λ_1 of Λ such that the first zeros of φ_n, $n\in\Lambda_1$ accumulate to some point λ_1, $|\lambda_1|\leqslant r$. Next, we arrange an infinite subset Λ_2 of Λ_1 such that the second zeros of φ_n, $n\in\Lambda_2$ accumulate to some point λ_2, $|\lambda_2|\leqslant r$. We continue this construction by induction and in a finite number of steps (notice that the number of the zeros in $\{z: |z|\leqslant r\}$ of each polynomial is uniformly bounded) we obtain an infinite subset Λ_k of Λ such that all zeros of φ_n in $\{z: |z|\leqslant r\}$ accumulate to $\lambda_1, \lambda_2, \ldots, \lambda_k$ as $n\to\infty$ along Λ.

Let $\rho>0$ be so small that the set

$$\{z: |z|<r\}\cap(\mathbb{C}\setminus conv(\sigma))$$

remains open and nonempty even after deletion of k closed disks $\{z: |z-\lambda_j| \leqslant 2\rho\}$, $j=1, ..., k$. We may also assume that ρ is so small that the open set

$$G_{\rho, K} = \{z: |z| < r, |z-\lambda_j| > 2\rho, j=1, ..., k\}$$

is connected and $G = G_{\rho, K} \cap (\mathbb{C} \setminus conv(\sigma)) \neq \varnothing$. Let us connect K with a point $z_0 \in G$ by a closed continuous path and denote by L the new compact obtained. By Theorem 8.3 the family $\{\psi_n/\varphi_n\}_{n \in \Lambda_k}$ is normal on L and by Lemma 9.1 it converges to $-F_\sigma$ uniformly at a neighborhood of z_0. Hence this family must converge to $-F_\sigma$ on K. ▮

Proposition 9.7. *Let σ satisfy* (L) *with parameters* (a, λ). *Then* (2.10) *holds uniformly on compact subsets of* $\mathbb{C} \setminus \bar{\lambda}\Delta_\alpha$, $\sin \alpha/2 = a$.

Proof. To simplify the notations we assume that $\lambda = 1$. Then by (2.9) $\mathscr{B}_0(\sigma) = \{-f_a\}$. It follows that $Z(\sigma) = \varnothing$. By Theorem 3 the zeros of the orthogonal polynomials associated with σ accumulate on Δ_α. Hence, by Lemma 8.5 for large n the Blaschke products b_n are uniformly bounded away from zero on every compact set of the form

$$K_{r, \varepsilon} = \{z: |z| \leqslant r\} \cup \{z: |z| \leqslant 1, |\arg z| \leqslant \alpha - \varepsilon\},$$

where $r < 1$, $\varepsilon > 0$. It follows that the family $\{b_n\}_{n \geqslant N(\varepsilon, r)}$ is normal on $K_{r, \varepsilon} \cup K^*_{r, \varepsilon}$, which obviously implies the result. ▮

9.3. *Examples.* We consider now some important examples, which demonstrate the possibilities covered by Theorem 9.6.

Example 1. Suppose that $\sigma \in \mathscr{P}$ satisfies the López condition. Lemma 2.3 shows that $\mathscr{B}_0(\sigma) = \{f_a(z)\}$. Since $f_a(z)$ does not vanish in $\mathbb{D}$, we obtain that $Z(\sigma) = \varnothing$, which is in complete argument with Theorem F.

Example 2. Let $\sigma \in \mathscr{P}$ satisfy

$$\lim_n |a_n| = 1,$$

$\{a_n\}_{n \geqslant 0}$ being the Geronimus parameters of σ. By Rakhmanov's lemma [41, Lemma 4] (see also [29, Corollary 9.5]) the measure σ is singular. By [29, Theorem 10] (see also [21, Theorem 5]) the derived set $(supp(\sigma))'$ is exactly the derived set of the sequence $\{-\bar{a}_n a_{n-1}\}_{n \geqslant 1}$. Clearly $\mathscr{B}_0(\sigma) = 1$, which implies that $Z(\sigma) = \varnothing$.

EXAMPLE 3. By [29, Theorem 9] any $\sigma \in \mathscr{P}$ satisfying

$$\underline{\lim}_n |a_n| > 0, \qquad \lim_n \arg(\bar{a}_n a_{n-1}) = \theta \tag{9.14}$$

has a gap in the spectrum centered at $\exp(i\theta)$. Let $\theta = 0$ for simplicity. Suppose that a sequence $\{a_n\}_{n \geqslant 0}$ of positive numbers in $(0, 1)$ is chosen so that

$$\lim_k a_{2k} = 1 \tag{9.15}$$

$$\{a_{2k+1}\}_{k \geqslant 0} \text{ is dense in } [\delta, 1], \qquad \delta > 0. \tag{9.16}$$

Then such a measure satisfies (9.14) with $\theta = 0$ and it is a singular measure by Rakhmanov's lemma. By (9.15) the domain $\mathscr{D}(x)$ of $x = \mathscr{S}f$, $f \in \mathscr{B}_0(\sigma)$, is either $\{0\}$ or $\{0, 1\}$. We obviously have $f \equiv 1$ in the first case and

$$f(z) = \frac{z+a}{1+az}, \qquad a \in [\delta, 1]$$

in the second. It follows that $Z(\sigma) = (-1, -\delta]$. Notice that $-1 \in (supp(\sigma))'$ and that $conv_0(\sigma)$ must contain $(-1, -\delta]$. By Theorem 9.6 the convergence in (9.9) takes place uniformly on compact subsets of $\mathbb{C} \setminus (supp(\sigma) \cup (-1, -\delta])$.

EXAMPLE 4. Let us consider a limit-periodic probability measure σ on $\mathbb{T}$, i.e., a measure with Geronimus parameters $\{a_n\}_{n \geqslant 0}$ satisfying

$$\lim_n a_{nk+j} = c_j, \qquad j = 0, 1, \ldots, k-1. \tag{9.17}$$

Assuming for simplicity that $c_j \in \mathbb{D}$, $j = 0, 1, \ldots, k-1$, we consider the limit measure $\bar{\sigma}$ with the periodic Geronimus parameters

$$(c_0, c_1, \ldots, c_{k-1}, c_0, c_1, \ldots, c_{k-1}, c_0, c_1, \ldots)$$

and the Galois dual measure $\tilde{\sigma}$ with the parameters

$$(-\bar{c}_{k-1}, -\bar{c}_{k-2}, \ldots, -\bar{c}_0, -\bar{c}_{k-1}, -\bar{c}_{k-2}, \ldots, -\bar{c}_0, \ldots).$$

The measures $\bar{\sigma}$ and $\tilde{\sigma}$ are dual in the sense that the limit set of the inverse Schur iterates for σ is finite and coincides with the finite set of *direct* Schur iterates of $\tilde{\sigma}$. It is clear that $\mathscr{B}_0(\sigma) = \mathscr{B}_0(\bar{\sigma})$ and that $0 \notin \mathscr{B}_0(\sigma)$ (if at least one $c_j \neq 0$). Hence $\sigma \in \mathrm{Mar}(\mathbb{T})$. This inclusion also follows by (1.28) of Theorem 1.8. Moreover, by [15, Theorem XI] $supp(\sigma) \notin \mathbb{T}$. Since the direct

Schur functions of $\tilde{\sigma}$ are algebraic, they may have only a finite number of zeros. It follows that $Z(\sigma)$ is finite.

EXAMPLE 5. The probability measure $\mu_{a,a'}$ with $a \neq a'$ is an example of measures considered in Example 4. Let us assume that $a < a' < 1$. Notice that $\mu_{a,a'}$ is limit-periodic by Theorem D. It is clear that $\mathscr{B}_0(\mu_{a,a'}) = \{-f_{a,a'}, -f_{a',a}\}$, where $f_{a,a'}$ and are defined by (4.21) and (4.23). It follows from (4.23) that $f_{a',a}$ does not vanish in $\mathbb{D}$. The function $f_{a,a'}$, on the other hand, has exactly one zero in $\mathbb{D}$: $z_0 = a/a'$. The point z_0 is a unique simple zero for the Schur function $-f_{a,a'} f_{a',a}$ of $\mu_{a,a'}$. It follows that for large n's every orthogonal polynomial has only one zero near $-a/a'$. The remaining zeros accumulate on the support $J(a, a') \cup \bar{J}(a, a')$ of $\mu_{a,a'}$.

ACKNOWLEDGMENT

I am grateful to Leonid Golinskii for turning my attention to the fact that ν_a is the equilibrium measure with respect to the logarithmic kernel for the closed arc Δ_α as well as for a very careful reading of the manuscript. I am also grateful to B. Simon for turning my attention to interesting papers [49–53].

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